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(CO)ENDS FOR REPRESENTATIONS OF TENSOR CATEGORIES

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Abstract. We generalize the notion of ends and coends in category theory to the realm of module categories over finite tensor categories. We call this new concept *module (co)end*. This tool allows us to give different proofs to several known results in the theory of representations of finite tensor categories. As a new application, we present a description of the relative Serre functor for module categories in terms of a module coend, in a analogous way as a Morita invariant description of the Nakayama functor of abelian categories presented in [4].

Introduction

Throughout this paper, $\Bbbk$ will denote a field, all categories will be finite (in the sense of [3]) abelian $\Bbbk$-linear categories, and all functors will be additive $\Bbbk$-linear. Given categories $\mathcal{M}, \mathcal{A}$, and a functor $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ the notion of the end $\int_{M\in\mathcal{M}} S$ and coend $\int^{M\in\mathcal{M}} S$ is a standard and very useful concept in category theory. The end of the functor S is an object $\int_{M\in\mathcal{M}} S \in \mathcal{A}$ together with dinatural transformations

$$\pi_M : \int_{M\in\mathcal{M}} S \dddot{\to} S(M, M)$$

with the following universal property; for any pair (B, d) consisting of an object $B \in \mathcal{A}$ and a dinatural transformation $d_M : B \dddot{\to} S(M, M)$, there exists a *unique* morphism $h : B \to \int_{M\in\mathcal{M}} S$ in $\mathcal{A}$ such that

$$d_M = \pi_M \circ h \quad \text{for any } M \in \mathcal{M}.$$

The notion of coend is defined dually.

If $\mathcal{M}$ is a finite abelian, $\Bbbk$-linear category, $\mathcal{M}$ can be thought of as a module category over $\mathrm{vect}_\Bbbk$, the tensor category of finite dimensional vector $\Bbbk$-spaces. If $\mathcal{M} = \mathfrak{m}_A$ is the category of finite dimensional right A-modules, where A is a finite dimensional $\Bbbk$-algebra, then $\mathcal{M}$ has a left $\mathrm{vect}_\Bbbk$-action

$$\mathrm{vect}_\Bbbk \times \mathfrak{m}_A \to \mathfrak{m}_A$$

$$(V, M) \mapsto V \otimes_\Bbbk M,$$

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where the right action on $V \otimes_{\Bbbk} M$ is given on the second tensorand. If $S : (\mathfrak{m}_A)^{\mathrm{op}} \times \mathfrak{m}_A \to \mathcal{A}$ is any functor, it posses a canonical natural isomorphism

$$\beta^V_{M,N} : S(M, V \otimes_{\Bbbk} N) \to S(V^* \otimes_{\Bbbk} M, N),$$

for any $V \in \operatorname{vect}_{\Bbbk}$, $M, N \in \mathfrak{m}_A$. See Proposition 3.14. The existence of β essentially follows from the additivity of the functor S.

If in addition $p_M : E \dddot{\to} S(M, M)$ is any dinatural transformation, it satisfies equation

$$S(\mathrm{ev}_V \otimes_{\Bbbk} \mathrm{id}_M, \mathrm{id}_M) p_M = S(m_{V^*,V,M}, \mathrm{id}_M) \beta^V_{V \otimes_{\Bbbk} M, M} p_{V \otimes_{\Bbbk} M}, \tag{0.1}$$

for any $V \in \operatorname{vect}_{\Bbbk}$. This is proven also in Proposition 3.14. This equation follows from the dinaturality of p. Here $\mathrm{ev}_V : V^* \otimes_{\Bbbk} V \to \Bbbk$ is the evaluation map, and $m_{W,V,M} : (W \otimes_{\Bbbk} V) \otimes_{\Bbbk} M \to W \otimes_{\Bbbk} (V \otimes_{\Bbbk} M)$ is the canonical associativity of vector spaces. This implies that the end of S is the universal object among all dinatural transformations that satisfy (0.1). A similar observation can be made for the coend. This is the starting point to generalize the notion of (co)end, where we will replace the category $\operatorname{vect}_{\Bbbk}$ with an arbitrary tensor category.

Let $\mathcal{C}$ be a tensor category, and $\mathcal{M}$ be a left $\mathcal{C}$-module category with action given by $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$. This action induces a right action of $\mathcal{C}$ on the opposite category $\mathcal{M}^{\mathrm{op}}$:

$$\blacktriangleleft : \mathcal{M}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}^{\mathrm{op}}.$$

$$\overline{M} \blacktriangleleft X = \overline{X^* \triangleright M},$$

Here $\overline{M}$ is the object M thought as an object in $\mathcal{M}^{\mathrm{op}}$. Assume $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ is a functor. We can produce then two functors:

$$S \circ (\mathrm{Id} \times \triangleright),\ S \circ (\blacktriangleleft \times \mathrm{Id}) : \mathcal{M}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{M} \to \mathcal{A}.$$

Assume there exists a natural isomorphism $\beta : S \circ (\mathrm{Id} \times \triangleright) \to S \circ (\blacktriangleleft \times \mathrm{Id})$, that is

$$\beta^X_{M,N} : S(M, X \triangleright N) \to S(X^* \triangleright M, N).$$

We call this isomorphism a *pre-balancing* of S. In this general case, the pre-balancing is an extra structure of the functor S. We define the *module end* of S to be an object $E \in \mathcal{A}$ that comes with dinatural transformations $\pi_M : E \dddot{\to} S(M, M)$ such that the equation

$$S(\mathrm{ev}_X \triangleright \mathrm{id}_M, \mathrm{id}_M) \pi_M = S(m_{X^*,X,M}, \mathrm{id}_M) \beta^X_{X \triangleright M, M} \pi_{X \triangleright M}, \tag{0.2}$$

is fulfilled, and it is universal among all objects in $\mathcal{A}$ with dinatural transformations that satisfy (0.2). Unlike the case $\mathcal{C} = \operatorname{vect}_{\Bbbk}$, it may happen that a dinatural transformation does not satisfy (0.2). We denote the module end as $\oint_{M \in \mathcal{M}} (S, \beta)$, or sometimes simply as $\oint_{M \in \mathcal{M}} S$ whenever the pre-balancing β is undertstood from the context.

An analogous definition can be made to define *module coend*, and also to define module ends and coends starting from *right* $\mathcal{C}$-module categories.

In Section 3 we introduce the module (co)ends, and we prove several results that extend known properties of (co)ends. We prove that, when the tensor category $\mathcal{C} = \operatorname{vect}_{\Bbbk}$ our definition coincides with the usual (co)ends. See Proposition 3.14. We also study what happens when we restrict the module (co)ends to a tensor subcategory. See Proposition 3.12.

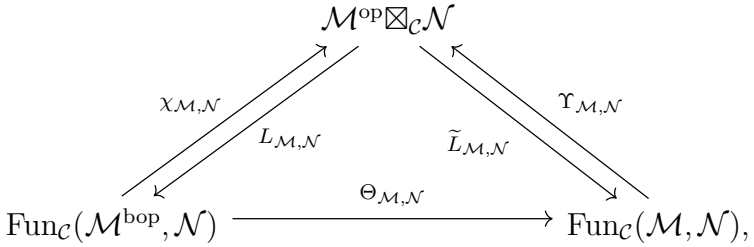
In Section 4 we give several applications. If $\mathcal{M}, \mathcal{N}$ are left $\mathcal{C}$-module categories, and $F, G : \mathcal{M} \to \mathcal{N}$ are $\mathcal{C}$-module functors, the functor

$$\operatorname{Hom}_{\mathcal{N}}(F(-), G(-)) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \operatorname{vect}_{\Bbbk}$$

has a canonical pre-balancing γ, and we prove that, there is an isomorphism

$$\operatorname{Nat}_m(F, G) \simeq \oint_{M \in \mathcal{M}} (\operatorname{Hom}_{\mathcal{N}}(F(-), G(-)), \gamma).$$

Here $\operatorname{Nat}_m(F, G)$ is the space of natural *module* transformations between F and G. See Proposition 4.2. Using this result we can set up a triangle of adjoint equivalences of categories



generalizing the triangle presented in [4]. Here it is required that $\mathcal{M}, \mathcal{N}$ are *exact* module categories. Here $\mathcal{M}^{\mathrm{op}}$ is the opposite category endowed with a *right* $\mathcal{C}$-action that comes from the action of $\mathcal{M}$ twisted by a (right) dual. Also $\mathcal{M}^{\mathrm{bop}} = (\mathcal{M}^{\mathrm{op}})^{\mathrm{op}}$. See Subsection 2.2 for these definitions. Observe that, $\mathcal{M}^{\mathrm{bop}} = \mathcal{M}$ as categories, but as a $\mathcal{C}$-module category $\mathcal{M}^{\mathrm{bop}}$ has the deformed action of $\mathcal{M}$ by a double dual.

The equivalences presented above are:

$$L_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \operatorname{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N}),$$

$$M \boxtimes_{\mathcal{C}} N \mapsto \underline{\operatorname{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-, M) \triangleright N,$$

$$\chi_{\mathcal{M},\mathcal{N}} : \operatorname{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N}) \to \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N},$$

$$F \mapsto \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} \overline{U} \boxtimes_{\mathcal{C}} F(\overline{\overline{U}}),$$

and on the other side of the triangle we have equivalences

$$\widetilde{L}_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \operatorname{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$$

$$M \boxtimes_{\mathcal{C}} N \mapsto \underline{\operatorname{Hom}}_{\mathcal{M}^{\mathrm{op}}}(M, -)^* \triangleright N,$$

$$\Upsilon_{\mathcal{M},\mathcal{N}} : \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N}) \to \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N},$$
$$F \mapsto \oint_{M\in\mathcal{M}} \overline{M} \boxtimes_{\mathcal{C}} F(M).$$

Here $\Theta_{\mathcal{M},\mathcal{N}} = \widetilde{L}_{\mathcal{M},\mathcal{N}} \circ \chi_{\mathcal{M},\mathcal{N}}$. The proof that $\widetilde{L}$ and L are equivalences is given in Lemma 4.5. The explicit description of a quasi-inverse of $\widetilde{L}$ is given in Theorem 4.7.

As a consequence of these equivalences, in Corollary 4.8 we obtain a kind of Peter-Weyl theorem for the regular A-bimodule $A = {}_AA_A$; that is, if $A \in \mathcal{C}$ is an algebra such that, the module category $\mathcal{C}_A$ is exact, there is an isomorphism of A-bimodules:

$$A \simeq \oint_{M\in\mathcal{M}} {}^*M \otimes M.$$

We also prove that the module functor $\Theta_{\mathcal{M},\mathcal{M}^{\mathrm{bop}}}(\mathrm{Id}\,)$ is equivalent, as module functors, to the (right) relative Serre functor of $\mathcal{M}$. See Theorem 4.14. This description is an analogous form of the Morita invariant description of the Nakayama functor presented in [4].

If $\mathcal{C}$ and $\mathcal{D}$ are Morita-equivalent tensor categories, this means that there exists an invertible $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{B}$; we prove that the correspondence

$$\mathcal{M} \mapsto \mathrm{Fun}_{\mathcal{C}}(\mathcal{B},\mathcal{M}), \quad \mathcal{N} \mapsto \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}},\mathcal{N})$$

is in fact part of a 2-equivalence between the 2-categories of $\mathcal{C}$-module categories and $\mathcal{D}$-module categories. This result was proven in [3]. We show in Theorem 4.19 that, for any $\mathcal{D}$-module category $\mathcal{N}$, the functor

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}},\mathcal{N})) \to \mathcal{N}$$
$$H \mapsto \oint_{B\in\mathcal{B}} H(B)(B)$$

is an equivalence of $\mathcal{D}$-module categories.

In the last Section we show that, the functor $\Upsilon : (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \to \mathcal{C}$ defined as

$$\Upsilon(G) = \oint_{M\in\mathcal{M}} \underline{\mathrm{Hom}}(M, G(M))$$

is a quasi-inverse of the canonical functor

$$can : \mathcal{C} \to (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}, \quad can(X)(M) = X \triangleright M.$$

See Theorem 4.21.

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Preliminaries and Notation. We denote by $\mathrm{vect}_{\Bbbk}$ the category of finite-dimensional $\Bbbk$-vector spaces. If $\mathcal{M}, \mathcal{N}$ are categories, and $F : \mathcal{M} \to \mathcal{N}$ is a functor, we shall denote by $F^{r.a.}, F^{l.a.} : \mathcal{N} \to \mathcal{M}$ its right and left adjoint, respectively.

For any category $\mathcal{M}$, the opposite category will be denoted by $\mathcal{M}^{\mathrm{op}}$. We shall denote by $\overline{M}, \overline{f}$ objects and morphisms in $\mathcal{M}^{\mathrm{op}}$ that correspond to M and f. We shall also denote by $F^{\mathrm{op}} : \mathcal{M}^{\mathrm{op}} \to \mathcal{N}^{\mathrm{op}}$ the opposite functor to F; that is, the functor defined as $F^{\mathrm{op}}(\overline{M}) = \overline{F(M)}$, $F^{\mathrm{op}}(\overline{f}) = \overline{F(f)}$ for any object M and any morphism f.

1. Finite tensor categories

For basic notions on finite tensor categories we refer to [2], [3]. Let $\mathcal{C}$ be a tensor category over $\Bbbk$; that is a rigid monoidal category with simple unit object $\mathbf{1}$.

If $\mathcal{C}$ has associativity constraint given by

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

we shall denote by $\mathcal{C}^{\mathrm{rev}}$, the tensor category whose underlying abelian category is $\mathcal{C}$, with *reverse* monoidal product

$$\otimes^{\mathrm{rev}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad X \otimes^{\mathrm{rev}} Y = Y \otimes X,$$

and associativity constraints

$$a^{\mathrm{rev}}_{X,Y,Z} : (X \otimes^{\mathrm{rev}} Y) \otimes^{\mathrm{rev}} Z \to X \otimes^{\mathrm{rev}} (Y \otimes^{\mathrm{rev}} Z),$$
$$a^{\mathrm{rev}}_{X,Y,Z} := a^{-1}_{Z,Y,X},$$

for any $X, Y, Z \in \mathcal{C}$. It is well known that for any pair of objects $X, Y \in \mathcal{C}$ there are canonical isomorphisms

$$\begin{aligned} \phi^r_{X,Y} &: (X \otimes Y)^* \to Y^* \otimes X^*, \\ \phi^l_{X,Y} &: {}^*(X \otimes Y) \to {}^*Y \otimes {}^*X. \end{aligned} \tag{1.1}$$

For any $X \in \mathcal{C}$ we shall denote by

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}, \quad \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$$

the evaluation and coevaluation. Abusing of the notation, we shall also denote by

$$\mathrm{ev}_X : X \otimes {}^*X \to \mathbf{1}, \quad \mathrm{coev}_X : \mathbf{1} \to {}^*X \otimes X$$

the evaluation and coevaluation for the left duals. If $f : X \to Y$ is an isomorphism in $\mathcal{C}$ then

$$\mathrm{ev}_Y(f \otimes \mathrm{id}_Y) = \mathrm{ev}_X(\mathrm{id}_X \otimes {}^*f). \tag{1.2}$$

For any $X, Y \in \mathcal{C}$ the following identities hold

$$\begin{aligned} \mathrm{ev}_{X \otimes Y} &= \mathrm{ev}_X(\mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{{}^*X})(\mathrm{id}_{X \otimes Y} \otimes \phi^l_{X,Y}), \\ (\phi^l_{X,Y} \otimes \mathrm{id}_{X \otimes Y})\mathrm{coev}_{X \otimes Y} &= (\mathrm{id}_{{}^*Y} \otimes \mathrm{coev}_X \otimes \mathrm{id}_Y)\mathrm{coev}_Y. \end{aligned} \tag{1.3}$$

Off course that similar identities hold for the right duals, but they won't be needed.

1.1. Algebras in tensor categories. In this subsection we assume that $\mathcal{C}$ is a strict tensor category, this means in particular that the associativity constraints are the identities. Let $A, B \in \mathcal{C}$ be algebras. We shall denote by

$$\mathcal{C}_A,\ {}_A\mathcal{C},\ {}_A\mathcal{C}_B$$

the categories of right A-modules, left A-modules and (A,B)-bimodules in $\mathcal{C}$, respectively. If $V \in \mathcal{C}_A$ is a right A-module with action given by $\rho_V : V\otimes A \to V$, and $W \in {}_A\mathcal{C}$ is a left A-module with action given by $\lambda_W : A\otimes W \to W$, we shall denote by $\pi^A_{V,W} : V\otimes W \to V\otimes_A W$ the coequalizer of the maps

$$\rho_V\otimes \mathrm{id}_W,\ \mathrm{id}_V\otimes\lambda_W : (V\otimes A)\otimes W \longrightarrow V\otimes W.$$

An object in the category ${}_A\mathcal{C}_B$ will be denoted as $(V, \lambda_V, \rho_V) \in {}_A\mathcal{C}_B$, where $\lambda_V : A\otimes V \to V$ is the left action, and $\rho_V : V\otimes B \to V$ is the right action. Since the tensor product is exact in both variables, then

$$\pi^A_{V,W\otimes U} = \pi^A_{V,W}\otimes \mathrm{id}_U,$$

for any $V \in \mathcal{C}_A, W \in {}_A\mathcal{C}$, $U \in \mathcal{C}$. We are going to freely use this fact without further mention.

1.2. Lemma. *Assume that $\mathcal{C}$ is a tensor category and $A, B \in \mathcal{C}$ are algebras. The following statements hold:*

(i) If $M \in \mathcal{C}_A$ then ${}^*M \in {}_A\mathcal{C}$.

(ii) There are natural isomorphisms

$$\mathrm{Hom}_B(M\otimes_A V, U) \simeq \mathrm{Hom}_{(A,B)}(V, {}^*M\otimes U), \tag{1.4}$$

$$\mathrm{Hom}_A(M, X\otimes N) \simeq \mathrm{Hom}_{\mathcal{C}}(M\otimes_A {}^*N, X), \tag{1.5}$$

$$\mathrm{Hom}_A(M, X\otimes N) \simeq \mathrm{Hom}_A(X^*\otimes M, N), \tag{1.6}$$

for any $X \in \mathcal{C}, M, N \in \mathcal{C}_A, V \in {}_A\mathcal{C}_B, U \in \mathcal{C}_B$.

Proof. (i). If $M \in \mathcal{C}_A$ then ${}^*M$ has structure of left A-module via $\lambda_{{}^*M} : A\otimes {}^*M \to {}^*M$ defined as

$$\lambda_{{}^*M} = (\mathrm{id}_{{}^*M}\otimes \mathrm{ev}_M)(\mathrm{id}_{{}^*M}\otimes\rho_M\otimes \mathrm{id}_{{}^*M})(\mathrm{coev}_M\otimes \mathrm{id}_{A\otimes {}^*M}). \tag{1.7}$$

(ii). Let us prove only the first isomorphism. The others follow similarly. The object $M\otimes_A V$ has a right B-module structure as follows. Consider $\phi : M\otimes V\otimes B \to M\otimes_A V$, $\phi = \pi_{M,V}(\mathrm{id}_M\otimes\rho_V)$. Then, $\rho_{M\otimes_A V} : M\otimes_A V\otimes B \to M\otimes_A V$ is defined as the unique morphism such that

$$\rho_{M\otimes_A V}(\pi_{M,V}\otimes \mathrm{id}_B) = \phi. \tag{1.8}$$

Define $\Phi : \mathrm{Hom}_B(M\otimes_A V, U) \to \mathrm{Hom}_{(A,B)}(V, {}^*M\otimes U)$ as

$$\Phi(f) = (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V})(\mathrm{coev}_M\otimes \mathrm{id}_V), \tag{1.9}$$

for any $f \in \mathrm{Hom}_B(M\otimes_A V, U)$. Let us show $\Phi(f)$ is a morphism of (A, B)-bimodules. We need to prove that

$$(\lambda_{{}^*M}\otimes \mathrm{id}_U)(\mathrm{id}_A\otimes\Phi(f)) = \Phi(f)\lambda_V, \tag{1.10}$$

and

$$(\mathrm{id}_{{}^*M}\otimes\rho_U)(\Phi(f)\otimes \mathrm{id}_B) = \Phi(f)\rho_V, \tag{1.11}$$

for any $(M, \rho_M) \in \mathcal{C}_A$, $(V, \lambda_V, \rho_V) \in {}_A\mathcal{C}_B$ and $(U, \rho_U) \in \mathcal{C}_B$. Here $\lambda_{{}^*M}$ is the left action of A on ${}^*M$ presented in (1.7).

The left hand side of (1.10) is equal to

$$\begin{aligned}
&(\lambda_{{}^*M}\otimes\mathrm{id}_U)(\mathrm{id}_A\otimes\Phi(f)) = \\
&\quad = (\mathrm{id}_{{}^*M}\otimes\mathrm{ev}_M\otimes\mathrm{id}_U)(\mathrm{id}_{{}^*M}\otimes\rho_M\otimes\mathrm{id}_{{}^*M\otimes U})(\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes{}^*M\otimes U})\\
&\quad (\mathrm{id}_{A\otimes{}^*M}\otimes f\pi_{M,V})(\mathrm{id}_A\otimes\mathrm{coev}_M\otimes\mathrm{id}_V)\\
&\quad = (\mathrm{id}_{{}^*M}\otimes\mathrm{ev}_M\otimes\mathrm{id}_U)(\mathrm{id}_{{}^*M}\otimes\rho_M\otimes\mathrm{id}_{{}^*M\otimes U})(\mathrm{id}_{{}^*M\otimes M\otimes A\otimes{}^*M}\otimes f\pi_{M,V})\\
&\quad (\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes{}^*M\otimes M\otimes V})(\mathrm{id}_A\otimes\mathrm{coev}_M\otimes\mathrm{id}_V)\\
&\quad = (\mathrm{id}_{{}^*M}\otimes\mathrm{ev}_M\otimes\mathrm{id}_V)(\mathrm{id}_{{}^*M\otimes M\otimes{}^*M}\otimes f\pi_{M,V})(\mathrm{id}_{{}^*M}\otimes\rho_M\otimes\mathrm{id}_{{}^*M\otimes M\otimes V})\\
&\quad (\mathrm{id}_{{}^*M\otimes M\otimes A}\otimes\mathrm{coev}_M\otimes\mathrm{id}_V)(\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes V})\\
&\quad = (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V})(\mathrm{id}_{{}^*M}\otimes\mathrm{ev}_M\otimes\mathrm{id}_{M\otimes V})(\mathrm{id}_{{}^*M\otimes M}\otimes\mathrm{coev}_M\otimes\mathrm{id}_V)\\
&\quad (\mathrm{id}_{{}^*M}\otimes\rho_M\otimes\mathrm{id}_V)(\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes V})\\
&\quad = (\mathrm{id}_{{}^*M}\otimes f)(\mathrm{id}_{{}^*M}\otimes\pi_{M,V}(\rho_M\otimes\mathrm{id}_V))(\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes V})\\
&\quad = (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V})(\mathrm{id}_{{}^*M\otimes M}\otimes\lambda_V)(\mathrm{coev}_M\otimes\mathrm{id}_{A\otimes V})\\
&\quad = (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V})(\mathrm{coev}_M\otimes\mathrm{id}_V)\lambda_V\\
&\quad = \Phi(f)\lambda_V.
\end{aligned}$$

The first equality is by the definition of $\lambda_{{}^*M}$ and $\Phi(f)$. The fifth equality follows from the rigidity axioms. The sixth equality is consequence of $\pi_{M,V}$ being the coequalizer of $\rho_M\otimes\mathrm{id}_V, \mathrm{id}_M\otimes\lambda_V$. The last equality follows by the definition of $\Phi(f)$.

Since f is a B-module morphism,

$$\rho_U(f\otimes\mathrm{id}_B) = f\rho_{M\otimes_A V}. \tag{1.12}$$

Using (1.8), this equation implies

$$\rho_U(f\pi_{M,V}\otimes\mathrm{id}_B) = f\rho_{M\otimes_A V}(\pi_{M,V}\otimes\mathrm{id}_B) = f\pi_{M,V}(\mathrm{id}_M\otimes\rho_V). \tag{1.13}$$

Let us prove (1.11). The left hand side of (1.11) is equal to

$$\begin{aligned}
(\mathrm{id}_{{}^*M}\otimes\rho_U)(\Phi(f)\otimes\mathrm{id}_B) &= (\mathrm{id}_{{}^*M}\otimes\rho_U(f\pi_{M,V}\otimes\mathrm{id}_B))(\mathrm{coev}_M\otimes\mathrm{id}_{V\otimes B})\\
&= (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V}(\mathrm{id}_M\otimes\rho_V))(\mathrm{coev}_M\otimes\mathrm{id}_{V\otimes B})\\
&= (\mathrm{id}_{{}^*M}\otimes f\pi_{M,V})(\mathrm{coev}_M\otimes\mathrm{id}_V)\rho_V\\
&= \Phi(f)\rho_V.
\end{aligned}$$

The first equality is by the definition of $\Phi(f)$. The second equality follows from (1.13). And the last equality again follows from the definition of $\Phi(f)$.

Now, let us show that Φ has an inverse. Let us define

$$\Psi : \mathrm{Hom}_{(A,B)}(V, {}^*M \otimes U) \to \mathrm{Hom}_B(M \otimes_A V, U)$$

as follows. Let $g \in \mathrm{Hom}_{(A,B)}(V, {}^*M \otimes U)$. Define $\Psi(g) = h$ where $h : M \otimes_A V \to U$ is the unique morphism such that

$$h\pi_{M,V} = (\mathrm{ev}_M \otimes \mathrm{id}_U)(\mathrm{id}_M \otimes g). \tag{1.14}$$

Let us show $\Psi(g)$ is a B-module morphism. That is

$$\rho_U(h \otimes \mathrm{id}_B) = h\rho_{M \otimes_A V}.$$

For this, it is enough to prove

$$\rho_U(h\pi_{M,V} \otimes \mathrm{id}_B) = h\rho_{M \otimes_A V}(\pi_{M,V} \otimes \mathrm{id}_B).$$

Starting from the left hand side

$$\begin{aligned}
\rho_U(h\pi_{M,V} \otimes \mathrm{id}_B) &= \rho_U(\mathrm{ev}_M \otimes \mathrm{id}_{U \otimes B})(\mathrm{id}_M \otimes g \otimes \mathrm{id}_B) \\
&= (\mathrm{ev}_M \otimes \mathrm{id}_U)(\mathrm{id}_{M \otimes {}^*M} \otimes \rho_U)(\mathrm{id}_M \otimes g \otimes \mathrm{id}_B) \\
&= (\mathrm{ev}_M \otimes \mathrm{id}_U)(\mathrm{id}_M \otimes g\rho_V) \\
&= h\pi_{M,V}(\mathrm{id}_M \otimes \rho_V) \\
&= h\rho_{M \otimes_A V}(\pi_{M,V} \otimes \mathrm{id}_B).
\end{aligned}$$

The first equality is by (1.14). The third equality is consequence of g being a B-module morphism. The fourth equality follows from (1.14) and the last equality follows from (1.8). Let us show Φ and Ψ are inverses of each another. Let be $f \in \mathrm{Hom}_B(M \otimes_A V, U)$. We have

$$\Psi\Phi(f) = \Psi((\mathrm{id}_{{}^*M} \otimes f\pi_{M,V})(\mathrm{coev}_M \otimes \mathrm{id}_V)) = h$$

where

$$\begin{aligned}
h\pi_{M,V} &= (\mathrm{ev}_M \otimes \mathrm{id}_U)(\mathrm{id}_{{}^*M \otimes M} \otimes f\pi_{M,V})(\mathrm{id}_M \otimes \mathrm{coev}_M \mathrm{id}_V) \\
&= f\pi_{M,V}(\mathrm{ev}_M \otimes \mathrm{id}_{M \otimes V})(\mathrm{id}_M \otimes \mathrm{coev}_M \otimes \mathrm{id}_V) \\
&= f\pi_{M,V}
\end{aligned}$$

The first equality is the definition of h, and the last equality follows from the rigidity axioms. Therefore, $h = f$ and $\Psi\Phi(f) = f$. The proof of $\Phi\Psi = \mathrm{Id}$ follows similarly.

We shall only sketch the proof of isomorphism (1.5). Define

$$\begin{aligned}
\Phi^A_{M,X,N} : \mathrm{Hom}_A(M, X \otimes N) &\to \mathrm{Hom}_{\mathcal{C}}(M \otimes_A {}^*N, X), \\
\Phi^A_{M,X,N}(\alpha)\pi^A_{M,{}^*N} &= (\mathrm{id}_X \otimes \mathrm{ev}_N)(\alpha \otimes \mathrm{id}_{{}^*N}),
\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}\Psi^A_{M,X,N} : \mathrm{Hom}_{\mathcal{C}}(M\otimes_A {}^*N, X) &\to \mathrm{Hom}_A(M, X\otimes N),\\ \Psi^A_{M,X,N}(\alpha) &= (\alpha\pi^A_{M,{}^*N}\otimes \mathrm{id}_N)(\mathrm{id}_M\otimes \mathrm{coev}_N).\end{aligned} \tag{1.16}$$

It follows by a direct calculation that $\Phi^A_{M,X,N}$ and $\Psi^A_{M,X,N}$ are well-defined and they are one the inverse of the other. ∎

2. Representations of tensor categories

A left *module* category over $\mathcal{C}$ is a category $\mathcal{M}$ together with a $\Bbbk$-bilinear bifunctor $\triangleright : \mathcal{C}\times\mathcal{M}\to\mathcal{M}$, exact in each variable, endowed with natural associativity and unit isomorphisms

$$m_{X,Y,M} : (X\otimes Y)\triangleright M \to X\triangleright(Y\triangleright M), \quad \ell_M : \mathbf{1}\triangleright M \to M.$$

These isomorphisms are subject to the following conditions:

$$m_{X,Y,Z\triangleright M}\, m_{X\otimes Y,Z,M} = (\mathrm{id}_X \triangleright m_{Y,Z,M})\, m_{X,Y\otimes Z,M}(a_{X,Y,Z}\triangleright \mathrm{id}_M), \tag{2.1}$$

$$(\mathrm{id}_X \triangleright \ell_M)m_{X,\mathbf{1},M} = r_X \triangleright \mathrm{id}_M, \tag{2.2}$$

for any $X, Y, Z \in \mathcal{C}, M \in \mathcal{M}$. Here a is the associativity constraint of $\mathcal{C}$. Sometimes we shall also say that $\mathcal{M}$ is a *$\mathcal{C}$-module category* or a representation of $\mathcal{C}$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be a pair of $\mathcal{C}$-modules. A *module functor* is a pair (F, c), where $F : \mathcal{M}\to\mathcal{M}'$ is a functor equipped with natural isomorphisms

$$c_{X,M} : F(X\triangleright M)\to X\triangleright F(M),$$

$X\in\mathcal{C}$, $M\in\mathcal{M}$, such that for any $X, Y\in\mathcal{C}$, $M\in\mathcal{M}$:

$$(\mathrm{id}_X \triangleright c_{Y,M})c_{X,Y\triangleright M}F(m_{X,Y,M}) = m_{X,Y,F(M)}\, c_{X\otimes Y,M} \tag{2.3}$$

$$\ell_{F(M)}\, c_{\mathbf{1},M} = F(\ell_M). \tag{2.4}$$

There is a composition of module functors: if $\mathcal{M}''$ is a $\mathcal{C}$-module category and $(G, d) : \mathcal{M}'\to\mathcal{M}''$ is another module functor then the composition

$$(G\circ F, e) : \mathcal{M}\to\mathcal{M}'', \qquad e_{X,M} = d_{X,F(M)}\circ G(c_{X,M}), \tag{2.5}$$

is also a module functor.

A *natural module transformation* between module functors (F, c) and (G, d) is a natural transformation $\theta : F\to G$ such that

$$d_{X,M}\theta_{X\triangleright M} = (\mathrm{id}_X \triangleright \theta_M)c_{X,M}, \tag{2.6}$$

for any $X\in\mathcal{C}$, $M\in\mathcal{M}$. The vector space of natural module transformations will be denoted by $\mathrm{Nat}_m(F, G)$. Two module functors F, G are *equivalent* if there exists a natural module isomorphism $\theta : F\to G$. We denote by $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M}')$ the category whose objects are module functors (F, c) from $\mathcal{M}$ to $\mathcal{M}'$ and arrows module natural transformations.

Two $\mathcal{C}$-modules $\mathcal{M}$ and $\mathcal{M}'$ are *equivalent* if there exist module functors $F : \mathcal{M} \to \mathcal{M}'$, $G : \mathcal{M}' \to \mathcal{M}$, and natural module isomorphisms $\mathrm{Id}_{\mathcal{M}'} \to F \circ G$, $\mathrm{Id}_{\mathcal{M}} \to G \circ F$.

A module is *indecomposable* if it is not equivalent to a direct sum of two non trivial modules. Recall from [3], that a module $\mathcal{M}$ is *exact* if for any projective object $P \in \mathcal{C}$ the object $P \triangleright M$ is projective in $\mathcal{M}$, for all $M \in \mathcal{M}$. If $\mathcal{M}$ is an exact indecomposable module category over $\mathcal{C}$, the dual category $\mathcal{C}^*_{\mathcal{M}} = \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is a finite tensor category [3]. The tensor product is the composition of module functors.

A *right module category* over $\mathcal{C}$ is a finite category $\mathcal{M}$ equipped with an exact bifunctor $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and natural isomorphisms

$$\widetilde{m}_{M,X,Y} : M \triangleleft (X \otimes Y) \to (M \triangleleft X) \triangleleft Y, \quad r_M : M \triangleleft \mathbf{1} \to M$$

such that

$$\widetilde{m}_{M \triangleleft X,Y,Z}\, \widetilde{m}_{M,X,Y \otimes Z}(\mathrm{id}_M \triangleleft a_{X,Y,Z}) = (\widetilde{m}_{M,X,Y} \triangleleft \mathrm{id}_Z)\, \widetilde{m}_{M,X \otimes Y,Z}, \tag{2.7}$$

$$(r_M \triangleleft \mathrm{id}_X)\widetilde{m}_{M,\mathbf{1},X} = \mathrm{id}_M \triangleleft l_X. \tag{2.8}$$

If $\mathcal{M}, \mathcal{M}'$ are right $\mathcal{C}$-modules, a module functor from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (T, d) where $T : \mathcal{M} \to \mathcal{M}'$ is a functor and $d_{M,X} : T(M \triangleleft X) \to T(M) \triangleleft X$ are natural isomorphisms such that for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(d_{M,X} \otimes \mathrm{id}_Y) d_{M \triangleleft X,Y} T(m_{M,X,Y}) = m_{T(M),X,Y}\, d_{M,X \otimes Y}, \tag{2.9}$$

$$r_{T(M)}\, d_{M,\mathbf{1}} = T(r_M). \tag{2.10}$$

The next result is well-known. See for example [1, Corollary 2.13.], [10, Prop. 2.2.4].

2.1. Lemma. *Let $\mathcal{M}, \mathcal{N}$ be left $\mathcal{C}$-module categories, and $F, G : \mathcal{M} \to \mathcal{N}$ are $\mathcal{C}$-module functors.*

(i) The right and left adjoint of F, if they exist, have structure of $\mathcal{C}$-module functor.

(ii) If $F \simeq G$ as $\mathcal{C}$-module functors, then $F^{l.a} \simeq G^{l.a}$, $F^{r.a} \simeq G^{r.a}$ as $\mathcal{C}$-module functors.

(iii) If F_1, F_2 are composable $\mathcal{C}$-module functors, there exists an isomorphism of $\mathcal{C}$-module functors

$$(F_1 \circ F_2)^{l.a} \simeq F_2^{l.a} \circ F_1^{l.a},\ (F_1 \circ F_2)^{r.a} \simeq F_2^{r.a} \circ F_1^{r.a}.$$

■

2.2. Bimodule categories. Assume that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are tensor categories. A $(\mathcal{C}, \mathcal{D})-$ *bimodule category* is a category $\mathcal{M}$ with left $\mathcal{C}$-module category structure $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, and right $\mathcal{D}$-module category structure $\triangleleft : \mathcal{M} \times \mathcal{D} \to \mathcal{M}$, equipped with natural isomorphisms

$$\{\gamma_{X,M,Y} : (X \triangleright M) \triangleleft Y \to X \triangleright (M \triangleleft Y), X \in \mathcal{C}, Y \in \mathcal{D}, M \in \mathcal{M}\}$$

satisfying certain axioms. For details the reader is referred to [7], [8].

If $\mathcal{M}$ is a right $\mathcal{C}$-module category then the opposite category $\mathcal{M}^{\mathrm{op}}$ has a left $\mathcal{C}$-action given by

$$\mathcal{C} \times \mathcal{M}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}},$$
$$(X, M) \mapsto M \triangleleft X^*,$$

and associativity isomorphisms $m^{\mathrm{op}}_{X,Y,M} = m_{M,Y^*,X^*}(\mathrm{id}_M \triangleleft \phi^r_{X,Y})$. Analogously, if $\mathcal{M}$ is a left $\mathcal{C}$-module category then $\mathcal{M}^{\mathrm{op}}$ has structure of right $\mathcal{C}$-module category, with action given by

$$\mathcal{M}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}^{\mathrm{op}},$$
$$(M, X) \mapsto X^* \triangleright M,$$

with associativity constraints $m^{\mathrm{op}}_{M,X,Y} = m_{Y^*,X^*,M}(\phi^r_{X,Y} \triangleright \mathrm{id}_M)$ for all $X, Y \in \mathcal{C}, M \in \mathcal{M}$. If $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category then $\mathcal{M}^{\mathrm{op}}$ is a $(\mathcal{D}, \mathcal{C})$-bimodule category.

If $\mathcal{M}$ is a left $\mathcal{C}$-module category, we shall denote by $\mathcal{M}^{\mathrm{bop}} = (\mathcal{M}^{\mathrm{op}})^{\mathrm{op}}$. That is, $\mathcal{M}^{\mathrm{bop}} = \mathcal{M}$ as categories, but the left action of $\mathcal{C}$ on $\mathcal{M}^{\mathrm{bop}}$ is

$$\blacktriangleright : \mathcal{C} \times \mathcal{M}^{\mathrm{bop}} \to \mathcal{M}^{\mathrm{bop}},$$
$$X \blacktriangleright M = X^{**} \triangleright M,$$

for any $X \in \mathcal{C}$, $M \in \mathcal{M}$.

2.3. Remark. *There is no problem to define the actions on the category $\mathcal{M}^{\mathrm{op}}$ using left duals instead of right duals. Our choice of using right duals is related to the choice of functors $L, \widetilde{L}$ presented later in* (4.4), (4.5).

Assume that $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category, and $\mathcal{N}$ is a $(\mathcal{C}, \mathcal{E})$-bimodule category. The category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ has a structure of $(\mathcal{D}, \mathcal{E})$-bimodule category. Let us briefly describe this structure. For more details, the reader is referred to [7]. The left and right actions are given by

$$\triangleright : \mathcal{D} \times \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}),$$
$$\triangleleft : \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \times \mathcal{E} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}),$$

where

$$(X \triangleright F)(M) = F(M \triangleleft X), \quad (F \triangleleft Y)(M) = F(M) \triangleleft Y, \tag{2.11}$$

for any $X \in \mathcal{D}$, $Y \in \mathcal{E}$, $F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ and $M \in \mathcal{M}$.

2.4. The internal Hom. Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ be a left $\mathcal{C}$-module category. For any pair of objects $M, N \in \mathcal{M}$, the *internal Hom* is an object $\underline{\mathrm{Hom}}(M, N) \in \mathcal{C}$ representing the left exact functor

$$\mathrm{Hom}_{\mathcal{M}}(- \triangleright M, N) : \mathcal{C}^{\mathrm{op}} \to \mathrm{vect}_{\Bbbk}.$$

This means that, there are natural isomorphisms, one the inverse of each other,

$$\begin{aligned} \phi^X_{M,N} &: \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N)) \to \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N), \\ \psi^X_{M,N} &: \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \to \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N)), \end{aligned} \tag{2.12}$$

for all $M, N \in \mathcal{M}$, $X \in \mathcal{C}$. Sometimes we shall denote the internal Hom of the module category $\mathcal{M}$ by $\underline{\mathrm{Hom}}_{\mathcal{M}}$ to emphasize that it is related to this module category. Similarly, if $\mathcal{N}$ is a right $\mathcal{C}$-module category, for any pair $M, N \in \mathcal{N}$ the internal hom is the object $\underline{\mathrm{Hom}}(M, N) \in \mathcal{C}$ representing the left exact functor

$$\mathrm{Hom}_{\mathcal{M}}(M \triangleleft -, N) : \mathcal{C}^{\mathrm{op}} \to \mathrm{vect}_{\Bbbk}.$$

2.5. Lemma. *The following statements hold.*

1. *Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. There are natural isomorphisms*

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(X \triangleright M, N) \simeq \underline{\mathrm{Hom}}_{\mathcal{M}}(M, N) \otimes X^*,$$

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(M, X \triangleright N) \simeq X \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}(M, N).$$

   for any $M, N \in \mathcal{M}$, $X \in \mathcal{C}$.

2. *Analogously, if $\mathcal{N}$ is a right $\mathcal{C}$-module category, there are natural isomorphisms*

$$\underline{\mathrm{Hom}}_{\mathcal{N}}(M \triangleleft X, N) \simeq {}^*X \otimes \underline{\mathrm{Hom}}_{\mathcal{N}}(M, N),$$

$$\underline{\mathrm{Hom}}_{\mathcal{N}}(M, N \triangleleft X) \simeq \underline{\mathrm{Hom}}_{\mathcal{N}}(M, N) \otimes X.$$

   for any $M, N \in \mathcal{N}$, $X \in \mathcal{C}$.

Proof. The functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \to \mathcal{C}$ is the right adjoint of the functor $R_M : \mathcal{C} \to \mathcal{M}$, $R_M(X) = X \triangleright M$. Since R_M is a $\mathcal{C}$-module functor then, it follows from Lemma 2.1 that, $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -)$ is also a $\mathcal{C}$-module functor. This implies in particular that there are natural isomorphisms

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(M, X \triangleright N) \simeq X \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}(M, N).$$

The other three isomorphisms follow in a similar way. ∎

Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. There is a relation between the internal hom of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{op}}$, stated in the next Lemma.

2.6. Lemma. *For any $M \in \mathcal{M}$, the functors*

$$^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-),\ \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-,\overline{M}) : \mathcal{M}^{\mathrm{bop}} \to \mathcal{C},$$

are equivalent $\mathcal{C}$-module functors. Also the functors

$$\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(M,-)^*,\ {}^*\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M) : \mathcal{M} \to \mathcal{C}$$

are equivalent $\mathcal{C}$-module functors. In particular, there are natural isomorphisms

$$^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,N) \simeq \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(\overline{N},\overline{M}),$$

for any $M, N \in \mathcal{M}$.

Proof. The functors $D : \mathcal{C} \to \mathcal{C}^{\mathrm{bop}}$, $D(X) = X^{**}$, and $L_M : \mathcal{C}^{\mathrm{bop}} \to \mathcal{M}^{\mathrm{bop}}$, $L_M(X) = X \triangleright M$, are $\mathcal{C}$-module functors. A straightforward computation shows that

$$(L_M \circ D)^{r.a.} \simeq \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-,\overline{M}),$$

$$D^{r.a.} \simeq {}^{**}(-), \quad (L_M)^{r.a.} \simeq \underline{\mathrm{Hom}}_{\mathcal{M}}(M,-).$$

Since D and L_M are $\mathcal{C}$-module functors, then, using Lemma 2.1 (i), it follows that, functors $^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-),\ \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-,\overline{M}) : \mathcal{M}^{\mathrm{bop}} \to \mathcal{C}$, are $\mathcal{C}$-module functors. Since $(L_M \circ D)^{r.a.} \simeq D^{r.a.} \circ (L_M)^{r.a.}$, it follows from Lemma 2.1 (iii) that, functors

$$^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-),\ \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-,\overline{M}) : \mathcal{M}^{\mathrm{bop}} \to \mathcal{C},$$

are equivalent as $\mathcal{C}$-module functors. The proof that functors

$$\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(M,-)^*,\ {}^*\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M) : \mathcal{M} \to \mathcal{C}$$

are equivalent is done by showing that both functors are left adjoint of $L_M : \mathcal{C} \to \mathcal{M}, L_M(X) = X \triangleright M$. ∎

2.7. Proposition. *Let $A \in \mathcal{C}$ be an algebra. The following statements hold.*

(i) For any $M, N \in \mathcal{C}_A$, $\underline{\mathrm{Hom}}_{\mathcal{C}_A}(M,N) = (M \otimes_A {}^*N)^*$.

(ii) For any $M, N \in \mathcal{C}_A$, $\underline{\mathrm{Hom}}_{(\mathcal{C}_A)^{\mathrm{op}}}(M,N) = {}^*(N \otimes_A {}^*M)$.

Proof. Both calculations of the internal hom follow from (1.5). ∎

The following result is [6, Lemma 3]. We include the proof since we will need later an explicit description of certain isomorphism.

2.8. Lemma. *Let $\mathcal{M}$ be an exact module category over $\mathcal{C}$, and $F : \mathcal{M} \to \mathcal{M}$ be a $\mathcal{C}$-module functor with left adjoint $F^{l.a.} : \mathcal{M} \to \mathcal{M}$. Then, there are natural isomorphisms*

$$\xi_{M,N} : \underline{\mathrm{Hom}}(M, F(N)) \to \underline{\mathrm{Hom}}(F^{l.a.}(M), N).$$

Proof. Since F is a module functor, then $F^{l.a.}$ is also a module functor. Let us denote by

$$b_{X,M} : F^{l.a.}(X \triangleright M) \to X \triangleright F^{l.a.}(M)$$

its module structure. Let $\Omega_{M,N} : \mathrm{Hom}_{\mathcal{M}}(M, F(N)) \to \mathrm{Hom}_{\mathcal{M}}(F^{l.a.}(M), N)$ be natural isomorphisms. Take $X \in \mathcal{C}$. The desired natural isomorphism is the one induced by the composition of isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, F(N))) \simeq \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, F(N)) \\
&\simeq \mathrm{Hom}_{\mathcal{M}}(F^{l.a.}(X \triangleright M), N) \simeq \mathrm{Hom}_{\mathcal{M}}(X \triangleright F^{l.a.}(M), N) \simeq \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(F^{l.a.}(M), N).
\end{aligned}$$

Using isomorphisms (2.12), one can describe explicitly this isomorphism as

$$\xi_{M,N} = \psi^{Z}_{F^{l.a.}(M),N}\big(\Omega_{Z \triangleright M,N}(\phi^{Z}_{M,F(N)}(\mathrm{id}_Z))b^{-1}_{Z,M}\big), \tag{2.13}$$

where $Z = \underline{\mathrm{Hom}}(M, F(N))$. ∎

2.9. The relative Serre functor. Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. Following [11], [5] we recall the definition of the relative Serre functor of a module category. The reader is also referred to [15].

2.10. Definition. *A relative Serre functor for $\mathcal{M}$ is a pair $(\mathbb{S}_{\mathcal{M}}, \phi)$, where $\mathbb{S}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a functor equipped with natural isomorphisms*

$$\phi_{M,N} : \underline{\mathrm{Hom}}(M, N)^* \simeq \underline{\mathrm{Hom}}(N, \mathbb{S}_{\mathcal{M}}(M)), \tag{2.14}$$

for any $M, N \in \mathcal{M}$.

In the next Proposition we summarize some known facts about relative Serre functors that will be used later.

2.11. Proposition. *Let $\mathcal{M}$ be a left module category over $\mathcal{C}$. The following holds.*

(i) $\mathcal{M}$ posses a relative Serre functor if and only if $\mathcal{M}$ is exact.

(ii) The functor $\mathbb{S}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^{\mathrm{bop}}$ is an equivalence of $\mathcal{C}$-module categories.

(iii) The natural isomorphism $\phi_{M,N} : \underline{\mathrm{Hom}}(M, N)^* \to \underline{\mathrm{Hom}}(N, \mathbb{S}_{\mathcal{M}}(M))$, is an isomorphism of $\mathcal{C}$-bimodule functors.

(iv) The relative Serre functor is unique up to isomorphism of $\mathcal{C}$-module functors.

■

2.12. Balanced tensor functors and Deligne tensor product. We shall briefly recall the definition of the *relative Deligne tensor product* over a tensor category. The reader is referred to [1], [7] for more details. Assume that $\mathcal{M}$ is a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category. Let $\mathcal{A}$ be a category.

A *$\mathcal{C}$-balanced functor* is a pair (Φ, b), where $\Phi : \mathcal{M}\times\mathcal{N} \to \mathcal{A}$ is a functor, right exact in each variable, equipped with natural isomorphisms $b_{M,X,N} : \Phi(M \triangleleft X, N) \to \Phi(M, X \triangleright N)$ such that it satisfies the pentagon

$$\Phi(\mathrm{id}_M, m^{\mathcal{N}}_{X,Y,N})b_{M,X\otimes Y,N} = b_{M,X,Y\triangleright N}b_{M\triangleleft X,Y,N}\Phi(m^{\mathcal{M}}_{M,X,Y}, \mathrm{id}_N), \tag{2.15}$$

for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$. The natural isomorphism b is called *the balancing* of Φ. If $(\Phi, b), (\widetilde{\Phi}, \widetilde{b}) : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ are $\mathcal{C}$-balanced functors, a *$\mathcal{C}$-balanced natural transformation* $\alpha : \Phi \to \widetilde{\Phi}$ is a natural transformation such that

$$\alpha_{M,X\triangleright N}b_{M,X,N} = \widetilde{b}_{M,X,N}\alpha_{M\triangleleft X,N}, \tag{2.16}$$

for any $X \in \mathcal{C}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$. The *balanced tensor product* (or sometimes called *relative Deligne tensor product*) is a category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$, equipped with a $\mathcal{C}$-balanced functor $\boxtimes_{\mathcal{C}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ such that for any category $\mathcal{A}$ the functor

$$\mathrm{Rex}\,(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A}) \to \mathrm{Bal}(\mathcal{M} \times \mathcal{N}, \mathcal{A})$$

$$F \mapsto F \circ \boxtimes_{\mathcal{C}}$$

is an equivalence of categories. Here $\mathrm{Bal}(\mathcal{M} \times \mathcal{N}, \mathcal{A})$ denotes the category of $\mathcal{C}$-balanced functors and $\mathcal{C}$-balanced natural transformations.

2.13. Lemma. *Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be right $\mathcal{C}$-module categories and $\mathcal{N}, \widetilde{\mathcal{N}}$ be left $\mathcal{C}$-module categories. If $(F, c) : \widetilde{\mathcal{M}} \to \mathcal{M}$, $(G, d) : \widetilde{\mathcal{N}} \to \mathcal{N}$ are right exact module functors, and $(\Phi, b) : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ is a $\mathcal{C}$-balanced functor, then $\Phi \circ (F \times G) : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{N}} \to \mathcal{A}$ is a $\mathcal{C}$-balanced functor with balancing given by*

$$e_{M,X,N} = \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,N})b_{F(M),X,G(N)}\Phi(c_{M,X}, \mathrm{id}_{G(N)}), \tag{2.17}$$

for any $M \in \widetilde{\mathcal{M}}, N \in \widetilde{\mathcal{N}}, X \in \mathcal{C}$.

Proof. We must show that e satisfies (2.15). In this case we have to prove

$$\Phi(\mathrm{id}_{F(M)}, G(m^{\widetilde{\mathcal{N}}}_{X,Y,N}))e_{M,X\otimes Y,N} = e_{M,X,Y\triangleright N}e_{M\triangleleft X,Y,N}\Phi(F(m^{\widetilde{\mathcal{M}}}_{M,X,Y}), \mathrm{id}_{G(N)}), \tag{2.18}$$

for any $X, Y \in \mathcal{C}$, $M \in \widetilde{\mathcal{M}}, N \in \widetilde{\mathcal{N}}$. The left hand side of (2.18) is equal to

$$\begin{aligned}
&= \Phi(\mathrm{id}_{F(M)}, G(m^{\widetilde{\mathcal{N}}}_{X,Y,N})d^{-1}_{X\otimes Y,N})b_{F(M),X\otimes Y,G(N)}\Phi(c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N}(\mathrm{id}_X \triangleright d^{-1}_{Y,N})m^{\mathcal{N}}_{X,Y,G(N)})b_{F(M),X\otimes Y,G(N)}\\
&\Phi(c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N})\Phi(\mathrm{id}_{F(M)}, \mathrm{id}_X \triangleright d^{-1}_{Y,N})b_{F(M),X,Y\triangleright G(N)}b_{F(M)\triangleleft X,Y,G(N)}\\
&\Phi(m^{\mathcal{M}}_{F(M),X,Y}, \mathrm{id}_{G(N)})\Phi(c_{M,X\otimes Y}, \mathrm{id}_{G(N)}).
\end{aligned}$$

The first equality is by the definition of e. The second equality is a consequence of (G,d) being a module functor, and the last equality is because (Φ,b) is a $\mathcal{C}$-balanced functor. The right hand side of (2.18) is equal to

$$\begin{aligned}
&= e_{M,X,Y\triangleright N}\Phi(\mathrm{id}_{F(M\triangleleft X)}, d^{-1}_{Y,N})b_{F(M\triangleleft X),Y,G(N)}\Phi(c_{M\triangleleft X,Y}F(m^{\widetilde{\mathcal{M}}}_{M,X,Y}), \mathrm{id}_{G(N)})\\
&= e_{M,X,Y\triangleright N}\Phi(\mathrm{id}_{F(M\triangleleft X)}, d^{-1}_{Y,N})b_{F(M\triangleleft X),Y,G(N)}\Phi((c^{-1}_{M,X}\triangleleft \mathrm{id}_Y)\\
&m^{\mathcal{M}}_{F(M),X,Y}c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N})b_{F(M),X,G(Y\triangleright N)}\Phi(c_{M,X}, \mathrm{id}_{G(Y\triangleright N)})\Phi(\mathrm{id}_{F(M\triangleleft X)}, d^{-1}_{Y,N})\\
&b_{F(M\triangleleft X),Y,G(N)}\Phi((c^{-1}_{M,X}\triangleleft \mathrm{id}_Y)m^{\mathcal{M}}_{F(M),X,Y}c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N})b_{F(M),X,G(Y\triangleright N)}\Phi(c_{M,X}, \mathrm{id}_{G(Y\triangleright N)})\Phi(\mathrm{id}_{F(M\triangleleft X)}, d^{-1}_{Y,N})\\
&\Phi(c^{-1}_{M,X}, \mathrm{id}_{Y\triangleright G(N)})b_{F(M)\triangleleft X,Y,G(N)}\Phi(m^{\mathcal{M}}_{F(M),X,Y}c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N})b_{F(M),X,G(Y\triangleright N)}\Phi(\mathrm{id}_{F(M\triangleleft X)}, d^{-1}_{Y,N})b_{F(M)\triangleleft X,Y,G(N)}\\
&\Phi(m^{\mathcal{M}}_{F(M),X,Y}c_{M,X\otimes Y}, \mathrm{id}_{G(N)})\\
&= \Phi(\mathrm{id}_{F(M)}, d^{-1}_{X,Y\triangleright N})\Phi(\mathrm{id}_{F(M)}, \mathrm{id}_X\triangleright d^{-1}_{Y,N})b_{F(M),X,Y\triangleright G(N)}b_{F(M)\triangleleft X,Y,G(N)}\\
&\Phi(m^{\mathcal{M}}_{F(M),X,Y}c_{M,X\otimes Y}, \mathrm{id}_{G(N)}).
\end{aligned}$$

The first and third equalities follow by the definition of e. The second equality follows since (F,c) is a module functor. The fourth equality is consequence of the naturality of b for $c_{M,X}$, and the sixth equality is the naturality of b for $d_{Y,N}$. Since both sides are equal, we get the result.

∎

The next result is well-known.

2.14. Proposition. *Let $A,B\in\mathcal{C}$ be algebras. Thus, the categories $\mathcal{C}_A,\mathcal{C}_B$ are left $\mathcal{C}$-module categories. The following assertions hold.*

(i) The functor ${}^*(-):(\mathcal{C}_A)^{\mathrm{op}}\to {}_A\mathcal{C}$ is an equivalence of right $\mathcal{C}$-module categories.

(ii) The restriction of the tensor product $\otimes : {}_A\mathcal{C}\times\mathcal{C}_B\to {}_A\mathcal{C}_B$ is a $\mathcal{C}$-balanced functor, and induces an equivalence of categories $\widehat{\otimes}:{}_A\mathcal{C}\boxtimes_{\mathcal{C}}\mathcal{C}_B\to {}_A\mathcal{C}_B$, such that

$$\widehat{\otimes}\circ\boxtimes_{\mathcal{C}}\simeq\otimes,$$

as $\mathcal{C}$-balanced functors.

(iii) Assume that $\mathcal{C}_A$ is an exact module category. The functor $R:{}_A\mathcal{C}_B\to\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B)$, $V\mapsto -\otimes_A V$ is an equivalence of categories.

Proof. (i) The duality functor ${}^*(-):(\mathcal{C}_A)^{\mathrm{op}}\to {}_A\mathcal{C}$ has structure of module functor with isomorphisms given by

$$\phi^l_{X^*,M}:{}^*(X^*\otimes M)\to {}^*M\otimes X,$$

for any $X \in \mathcal{C}$, $M \in \mathcal{C}_A$. Here ϕ^l is the natural isomorphisms described in (1.1). Note that we are omitting the canonical natural isomorphism ${}^*(X^*) \simeq X$. For (ii) see [1]. The proof of (iii) can be found for example in [9, Prop. 3.3]. Exactness of the module category $\mathcal{C}_A$ implies that $\otimes_A$ is biexact. This fact was used in [9] to prove that R is a category equivalence. ∎

3. The (co)end for module categories

Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ be a left $\mathcal{C}$-module category. Assume that $\mathcal{A}$ is a category and $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ a functor equipped with natural isomorphisms

$$\beta^X_{M,N} : S(M, X \triangleright N) \to S(X^* \triangleright M, N), \tag{3.1}$$

for any $X \in \mathcal{C}, M, N \in \mathcal{M}$. We shall say that β is a *pre-balancing* of the functor S.

3.1. Definition. *The module end of the pair (S, β) is an object $E \in \mathcal{A}$ equipped with dinatural transformations $\pi_M : E \ddot{\to} S(M, M)$ such that*

$$S(ev_X \triangleright \mathrm{id}_M, \mathrm{id}_M)\pi_M = S(m_{X^*,X,M}, \mathrm{id}_M)\beta^X_{X \triangleright M, M}\pi_{X \triangleright M}, \tag{3.2}$$

for any $X \in \mathcal{C}, M \in \mathcal{M}$, universal with this property. This means that, if $\widetilde{E} \in \mathcal{A}$ is another object with dinatural transformations $\xi_M : \widetilde{E} \ddot{\to} S(M, M)$, such that they verify (3.2), there exists a unique morphism $h : \widetilde{E} \to E$ such that $\xi_M = \pi_M \circ h$.

Sometimes we will denote the module end as $\oint_{M \in \mathcal{M}}(S, \beta)$, or simply as $\oint_{M \in \mathcal{M}} S$, when the pre-balancing β is understood from the context. The *module coend* of the pair (S, β) is defined dually. This is an object $C \in \mathcal{A}$ equipped with dinatural transformations $\pi_M : S(M, M) \ddot{\to} C$ such that

$$\pi_M = \pi_{X^* \triangleright M}\beta^X_{M, X^* \triangleright M} S(\mathrm{id}_M, m_{X,X^*,M})S(\mathrm{id}_M, \mathrm{coev}_X \triangleright \mathrm{id}_M), \tag{3.3}$$

for any $X \in \mathcal{C}, M \in \mathcal{M}$, universal with this property. This means that, if $\widetilde{C} \in \mathcal{A}$ is another object with dinatural transformations $\lambda_M : S(M, M) \ddot{\to} \widetilde{C}$ such that they satisfy (3.3), there exists a unique morphism $g : C \to \widetilde{C}$ such that $g \circ \pi_M = \lambda_M$. The module coend will be denoted $\oint^{M \in \mathcal{M}}(S, \beta)$, or simply as $\oint^{M \in \mathcal{M}} S$.

A similar definition can be made for *right* $\mathcal{C}$-module categories. Let $\mathcal{B}$ be a category, and $\mathcal{N}$ be a right $\mathcal{C}$-module category endowed with a functor $S : \mathcal{N}^{\mathrm{op}} \times \mathcal{N} \to \mathcal{B}$ with a *pre-balancing*

$$\gamma^X_{M,N} : S(M \triangleleft X, N) \to S(M, N \triangleleft {}^*X),$$

for any $M, N \in \mathcal{N}$, $X \in \mathcal{C}$.

3.2. Definition. *The module end for S is an object $E \in \mathcal{B}$ equipped with dinatural transformations $\lambda_N : E \dddot{\to} S(N,N)$ such that*

$$\lambda_N = S(\mathrm{id}_N, \mathrm{id}_N \triangleleft ev_X) S(\mathrm{id}_N, m^{-1}_{N,X,{}^*X}) \gamma^X_{N,N\triangleleft X} \lambda_{N\triangleleft X}, \tag{3.4}$$

for any $N \in \mathcal{N}$, $X \in \mathcal{C}$. We shall also denote this module end by $\oint_{N\in\mathcal{N}}(S,\gamma)$.

Similarly, the module coend is an object $C \in \mathcal{B}$ with dinatural transformations $\lambda_N : S(N,N) \dddot{\to} C$ such that

$$\lambda_N S(\mathrm{id}_N \triangleleft coev_X, \mathrm{id}_N) = \lambda_{N\triangleleft^* X} \gamma^X_{N\triangleleft^* X, N} S(m^{-1}_{N,{}^*X,X}, \mathrm{id}_N), \tag{3.5}$$

for any $N \in \mathcal{N}$, $X \in \mathcal{C}$. We shall also denote this module coend by $\oint^{N\in\mathcal{N}}(S,\gamma)$.

In the next Proposition we collect some results about module ends that generalize well-known results in the theory of (co)ends. The proofs follow the same lines as the ones in usual ends. For the sake of completeness we include them.

3.3. Proposition. *Assume that $\mathcal{M},\mathcal{N}$ are left $\mathcal{C}$-module categories, and $S, \widetilde{S} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ are functors equipped with pre-balancings*

$$\beta^X_{M,N} : S(M, X \triangleright N) \to S({}^*X \triangleright M, N),$$
$$\widetilde{\beta}^X_{M,N} : \widetilde{S}(M, X \triangleright N) \to \widetilde{S}({}^*X \triangleright M, N),$$

$X \in \mathcal{C}, M, N \in \mathcal{M}$. The following assertions holds

(i) Assume that the module ends $\oint_{M\in\mathcal{M}}(S,\beta), \oint_{M\in\mathcal{M}}(\widetilde{S},\widetilde{\beta})$ exist and have dinatural transformations $\pi, \widetilde{\pi}$, respectively. If $\gamma : S \to \widetilde{S}$ is a natural transformation such that

$$\widetilde{\beta}^X_{M,N} \gamma_{(M,X\triangleright N)} = \gamma_{({}^*X\triangleright M,N)} \beta^X_{M,N}, \tag{3.6}$$

then there exists a unique map $\widehat{\gamma} : \oint_{M\in\mathcal{M}}(S,\beta) \to \oint_{M\in\mathcal{M}}(\widetilde{S},\widetilde{\beta})$ such that $\widetilde{\pi}_M \widehat{\gamma} = \gamma_{(M,M)} \pi_M$ for any $M \in \mathcal{M}$. If γ is a natural isomorphism, then $\widehat{\gamma}$ is an isomorphism.

(ii) If the end $\oint_{M\in\mathcal{M}}(S,\beta)$ exists, then for any object $U \in \mathcal{A}$, the end $\oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-))$ exists, and there is an isomorphism

$$\oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-)) \simeq \mathrm{Hom}_{\mathcal{A}}(U, \oint_{M\in\mathcal{M}} (S,\beta)).$$

Moreover, if $\oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-))$ exists for any $U \in \mathcal{A}$, then the end $\oint_{M\in\mathcal{M}}(S,\beta)$ exists.

(iii) Assume $F : \mathcal{A} \to \mathcal{A}'$ is a right exact functor. Then, there is an isomorphism

$$F(\oint_{N\in\mathcal{N}} (S,\beta)) \simeq \oint_{N\in\mathcal{N}} (F\circ S, F(\beta)).$$

(iv) If $F : \mathcal{M} \to \mathcal{N}$ is an equivalence of $\mathcal{C}$-module categories, then there is an isomorphism

$$\oint_{N\in\mathcal{N}} S \simeq \oint_{M\in\mathcal{M}} S(F(-), F(-)).$$

PROOF. (i). For any $M \in \mathcal{N}$ define $\lambda_M : \oint_{N\in\mathcal{N}}(S,\beta) \to \widetilde{S}(M,M)$ as $\lambda_M = \gamma_{(M,M)}\pi_M$. It follow straightforward that λ is dinatural and since γ satisfies (3.6), then λ satisfies (3.2). By the universality of the module end, there exists a morphism $\widehat{\gamma} : \oint_{M\in\mathcal{M}}(S,\beta) \to \oint_{M\in\mathcal{M}}(\widetilde{S},\widetilde{\beta})$ such that $\widetilde{\pi}_M\widehat{\gamma} = \lambda_M = \gamma_{(M,M)}\pi_M$.

(ii). Let us assume that $\oint_{M\in\mathcal{M}}(S,\beta)$ exists, and has associated to it dinatural transformations $\pi_N : \oint_{M\in\mathcal{M}}(S,\beta) \to S(N,N)$. For any $U \in \mathcal{A}$, the pre-balancing for the functor $\mathrm{Hom}_{\mathcal{A}}(U, S(-,-))$ is defined as

$$\beta^U_{X,M,N} : \mathrm{Hom}_{\mathcal{A}}(U, S(M, X \triangleright N)) \to \mathrm{Hom}_{\mathcal{A}}(U, S(X^* \triangleright M, N)),$$

$$\beta^U_{X,M,N}(f) = \beta^X_{M,N} \circ f.$$

Also define

$$\pi^U_N : \mathrm{Hom}_{\mathcal{A}}(U, \oint_{M\in\mathcal{M}}(S,\beta)) \to \mathrm{Hom}_{\mathcal{A}}(U, S(N,N)),$$

$$\pi^U_N(f) = \pi_N \circ f.$$

It follows by a straightforward computation that, π^U is a dinatural transformation, and they satisfy (3.2) using β^U. It also follows easily that $\mathrm{Hom}_{\mathcal{A}}(U, \oint_{M\in\mathcal{M}}(S,\beta))$ together with π^U satisfy the universal property of the module end, thus

$$\oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-)) \simeq \mathrm{Hom}_{\mathcal{A}}(U, \oint_{M\in\mathcal{M}}(S,\beta)).$$

Now, let us assume that $\oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-))$ exists for any $U \in \mathcal{A}$. Using item (i), we can define a functor

$$F : \mathcal{A}^{\mathrm{op}} \to \mathrm{vect}_{\Bbbk},$$

$$F(U) = \oint_{M\in\mathcal{M}} \mathrm{Hom}_{\mathcal{A}}(U, S(-,-)).$$

We shall prove that F es left exact, and thus it is representable. The object representing the functor F will be a candidate for the module end $\oint_{M\in\mathcal{M}}(S,\beta)$.

For any $M \in \mathcal{M}$, and any $f : U \to V$ in $\mathcal{A}$, denote

$$(\alpha_f)_M : \mathrm{Hom}_{\mathcal{A}}(V, S(M,M)) \to \mathrm{Hom}_{\mathcal{A}}(U, S(M,M))$$

$$(\alpha_f)_M(g) = g \circ f.$$

To prove that F is left exact, we need to show that, for any morphism $f : U \to V$ in $\mathcal{A}$, $F(\mathrm{coKer}\,(f)) = \mathrm{Ker}\,(F(f))$. Let be $q = \mathrm{coKer}\,(f) : V \to C$, and $l : K \to F(V)$ be a $\Bbbk$-linear map such that $F(f) \circ l = 0$. Then

$$(\alpha_f)_M \circ \pi^V_M \circ l = \pi^U_M \circ F(f) \circ l = 0.$$

The second equality follows from item (i). Since $\ker(\alpha_f) = \alpha_q$, there exists a map

$$h_M : K \to \mathrm{Hom}_{\mathcal{A}}(C, S(M,M)))$$

such that $(\alpha_q)_M \circ h_M = \pi^V_M \circ l$. It is not difficult to prove that h is a dinatural transformation, and they satisfy (3.2) (using the isomorphisms β^C). By the universal property of the module end, there exists a morphism $\phi : K \to F(C)$ such that $h_M = \pi^C_M \circ \phi$. It follows from item (i) that

$$(\alpha_q)_M \circ \pi^C_M \circ \phi = \pi^V_M \circ F(q) \circ \phi.$$

But also

$$(\alpha_q)_M \circ \pi^C_M \circ \phi = (\alpha_q)_M \circ h_M = \pi^V_M \circ l,$$

whence $l = F(q) \circ \phi$ and therefore $F(q) = \ker(F(f))$. Hence F is represented by an object $E \in \mathcal{A}$; $F(U) = \mathrm{Hom}_{\mathcal{A}}(U, E)$. The maps $\delta_M : E \to S(M, M)$, $\delta_M = \pi^E_M(\mathrm{id}_E)$ are dinatural transformations, and they satisfy (3.2). It follows by a straightforward computation that E together with δ satisfy the universal property of the module end, thus $E \simeq \oint_{M\in\mathcal{M}}(S, \beta)$.

The proof of (iii) and (iv) is straightforward. ∎

3.4. Remark. *Off course that, similar results to those presented in Proposition 3.3 can be stated for module coends, and also for module (co)ends for right module categories.*

3.5. Relation between module (co)ends for right and left module categories. Let $\mathcal{A}$ be a category. Let $\mathcal{M}$ be a left $\mathcal{C}$-module category, and a functor $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ equipped with a pre-balancing $\beta^X_{M,N} : S(M, X \triangleright N) \to S(X^* \triangleright M, N)$. Then $\mathcal{N} = \mathcal{M}^{\mathrm{op}}$ is a right $\mathcal{C}$-module category. We can consider the functor

$$S^{\mathrm{op}} : \mathcal{N}^{\mathrm{op}} \times \mathcal{N} \to \mathcal{A}^{\mathrm{op}}.$$

It posses a pre-balancing

$$\gamma^X_{M,N} : S^{\mathrm{op}}(M \triangleleft X, N) \to S^{\mathrm{op}}(M, N \triangleleft {}^*X),$$

$$\gamma^X_{M,N} = \beta^X_{M,N}.$$

Note that, the pre-balancing γ is considered as a morphism in $\mathcal{A}^{\mathrm{op}}$. The next result is straightforward.

3.6. Lemma. *There are isomorphisms*

$$\overline{\oint_{M\in\mathcal{M}}(S, \beta)} \simeq \oint^{\overline{M}\in\mathcal{N}}(S^{\mathrm{op}}, \gamma),$$

$$\overline{\oint^{M\in\mathcal{M}}(S, \beta)} \simeq \oint_{\overline{M}\in\mathcal{N}}(S^{\mathrm{op}}, \gamma).$$

∎

A similar result also holds starting from a right $\mathcal{C}$-module category $\mathcal{N}$, and a functor $T : \mathcal{N}^{\mathrm{op}} \times \mathcal{N} \to \mathcal{A}$ equipped with a pre-balancing

$$\gamma^X_{M,N} : T(M \triangleleft X, N) \to T(M, N \triangleleft {}^*X).$$

If $\mathcal{M} = \mathcal{N}^{\mathrm{op}}$, then $\mathcal{M}$ is a left $\mathcal{C}$-module category, and we can consider the functor

$$T^{\mathrm{op}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}^{\mathrm{op}}$$

together with a pre-balancing

$$\beta^X_{M,N} : T^{\mathrm{op}}(M, X \triangleright N) \to T^{\mathrm{op}}(X^* \triangleright M, N)$$

$$\beta^X_{M,N} = \gamma^{X^{**}}_{M,N}.$$

The next result is a straightforward consequence of the definitions of module (co)end.

3.7. Lemma. *There are isomorphisms*

$$\overline{\oint_{N\in\mathcal{N}} (T,\gamma)} \simeq \oint^{\overline{M}\in\mathcal{M}} (T^{\mathrm{op}},\beta),$$

$$\overline{\oint^{M\in\mathcal{N}} (T,\gamma)} \simeq \oint_{\overline{M}\in\mathcal{M}} (T^{\mathrm{op}},\beta).$$

■

3.8. Parameter theorem for module ends. Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ a left $\mathcal{C}$-module category. Also, let $\mathcal{A}, \mathcal{B}$ be categories. We start with a functor $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{Fun}(\mathcal{A}, \mathcal{B})$ equipped with pre-balancing $\beta^X_{M,N} : S(M, X \triangleright N) \to S(X^* \triangleright M, N)$, for any $X \in \mathcal{C}, M, N \in \mathcal{M}$. If the end $\oint_{M\in\mathcal{M}}(S,\beta)$ exists, it is an object in the category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$; we denote this functor as

$$\big(\oint_{M\in\mathcal{M}} (S,\beta)\big)(-) : \mathcal{A} \to \mathcal{B}.$$

Alternatively, we can do the following construction. For any $A \in \mathcal{A}$ we get a functor $S_A : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{B}$, $S_A(M, N) = S(M, N)(A)$. This functor comes with a pre-balancing

$$\beta^A_{X,M,N} : S_A(M, X \triangleright N) \to S_A(X^* \triangleright M, N),$$

$$\beta^A_{X,M,N} = (\beta^X_{M,N})_A,$$

for any $X \in \mathcal{C}, M, N \in \mathcal{M}$. If the module end $\oint_{M\in\mathcal{M}}(S_A,\beta^A)$ exists, it is an object in $\mathcal{B}$, and it defines a functor $\mathfrak{S} : \mathcal{A} \to \mathcal{B}$. The proof of the next result follow straightforward.

3.9. THEOREM. *Provided all ends $\oint_{M\in\mathcal{M}}(S_A,\beta^A)$ exist, the functor $\mathfrak{S}$ has a canonical structure of module end for the functor S. We write*

$$\mathfrak{S}=\big(\oint_{M\in\mathcal{M}}(S,\beta)\big)(-).$$

■

3.10. REMARK. *Similar results can be obtained for module coends, and also for right $\mathcal{C}$-module categories.*

3.11. RESTRICTION OF MODULE (CO)ENDS TO TENSOR SUBCATEGORIES. In this Section, we shall show that the module (co)end coincides with the usual (co)end in the case the tensor category is $\mathrm{vect}_{\Bbbk}$. We also study what happens with the module (co)end when we restrict to a tensor subcategory.

Let $\mathcal{C}$ be a tensor category and $\mathcal{D}\subseteq\mathcal{C}$ be a tensor subcategory of $\mathcal{C}$. Assume also that $\mathcal{M}$ is a left $\mathcal{C}$-module category. We can consider the restricted $\mathcal{D}$-module category $\mathrm{Res}^{\mathcal{D}}_{\mathcal{C}}\mathcal{M}$. The next result is a straightforward consequence of the definition of module (co)ends.

3.12. PROPOSITION. *Let $S:\mathcal{M}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{A}$ be a functor equipped with pre-balancing $\beta^X_{M,N}:S(M,X\triangleright N)\to S(X^*\triangleright M,N)$.*

(i) There exists a monomorphism in $\mathcal{A}$

$$\oint_{M\in\mathcal{M}}(S,\beta)\hookrightarrow\oint_{M\in\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\mathcal{M}}(S,\beta).$$

(ii) There exists an epimorphism in $\mathcal{A}$

$$\oint^{M\in\mathrm{Res}^{\mathcal{D}}_{\mathcal{C}}\mathcal{M}}(S,\beta)\twoheadrightarrow\oint^{M\in\mathcal{M}}(S,\beta).$$

■

3.13. REMARK. *Similar result obtained in Proposition 3.12, is valid for right module categories.*

The next result says that, the module (co)end coincides with the usual one in the case $\mathcal{C}=\mathrm{vect}_{\Bbbk}$.

3.14. PROPOSITION. *Let $\mathcal{M},\mathcal{A}$ be categories, and $S:\mathcal{M}^{\mathrm{op}}\times\mathcal{M}\to\mathcal{A}$ be a functor. In particular $\mathcal{M}$ is a left $\mathrm{vect}_{\Bbbk}$-module category. The functor S has a canonical pre-balancing β such that there are isomorphisms*

$$\int_{M\in\mathcal{M}}S\simeq\oint_{M\in\mathcal{M}}(S,\beta),$$
$$\int^{M\in\mathcal{M}}S\simeq\oint^{M\in\mathcal{M}}(S,\beta).$$

PROOF. We shall prove the first isomorphism concerning the usual end and the module end. The other isomorphism for the coend follows similarly. For this, we will show that, for such a functor S there exists a canonical pre-balancing β such that *any* dinatural transformation $\pi_M : E \ddot{\to} S(M, M)$ satisfies (3.2).

Since $\mathcal{M}$ is a finite abelian $\Bbbk$-linear category, there exists a finite dimensional $\Bbbk$-algebra A such that, $\mathcal{M}$ is equivalent to the category of finite dimensional right A-modules $\mathfrak{m}_A$. The action of $\mathrm{vect}_\Bbbk$ on $\mathfrak{m}_A$ is

$$\triangleright : \mathrm{vect}_\Bbbk \times \mathfrak{m}_A \to \mathfrak{m}_A,$$

$$X \triangleright M = X \otimes_\Bbbk M,$$

for any $X \in \mathrm{vect}_\Bbbk$, $M \in \mathfrak{m}_A$. The right action of A on $X \otimes_\Bbbk M$ is on the second tensorand. For any $X, Y \in \mathrm{vect}_\Bbbk$, $M \in \mathfrak{m}_A$ the associativity of this module category is

$$m_{X,Y,M} : (X \otimes_\Bbbk Y) \otimes_\Bbbk M \to X \otimes_\Bbbk (Y \otimes_\Bbbk M),$$

$$m_{X,Y,M}((x \otimes y) \otimes m) = x \otimes (y \otimes m).$$

For any $X \in \mathrm{vect}_\Bbbk$, $x \in X$, we denote by $\delta_x : X \to \Bbbk$ the unique linear transformation that sends x to 1, and any element of a direct complement of $< x >$ to 0. If $M \in \mathfrak{m}_A$, $X \in \mathrm{vect}_\Bbbk$, $x \in X$ we shall denote by

$$\iota_x^M : M \to X \triangleright M, \quad p_x^M : X \triangleright M \to M,$$

$$\iota_x^M(m) = x \otimes m, \quad p_x^M(y \otimes m) = \delta_x(y)m,$$

for any $y \in X, m \in M$. Let $(x_i), (f_i)$ be a pair of dual basis of X and X^* respectively. For any $x \in X, f \in X^*$ it is easy to verify that

$$\sum_i \delta_{x_i}(x)\delta_{f_i}(f) = f(x).$$

This equality implies that

$$\mathrm{ev}_X \otimes \mathrm{id}_M = \sum_i p_{x_i}^M p_{f_i}^{X \otimes_\Bbbk M} m_{X^*,X,M}. \tag{3.7}$$

Also, one can verify that

$$\sum_i \iota_{x_i}^M p_{x_i}^M = \mathrm{id}_{X \otimes_\Bbbk M}, \quad p_x^M \iota_y^M = \delta_x(y)\mathrm{id}_M. \tag{3.8}$$

For any $M, N \in \mathfrak{m}_A$ let us denote

$$\beta_{M,N}^X : S(M, X \triangleright N) \to S(X^* \triangleright M, N),$$

$$\beta_{M,N}^X = \oplus_i S(p_{f_i}^M, p_{x_i}^N),$$

where $(x_i), (f_i)$ is a pair of dual basis of X and X^*, respectively. Using (3.8), one can check that, $\beta^X_{M,N}$ is an isomorphism with inverse

$$\oplus_i S(\iota^M_{f_i}, \iota^N_{x_i}) : S(X^* \triangleright M, N) \to S(M, X \triangleright N).$$

Let $E \in \mathcal{A}$ be an object and $\pi_M : E \dddot{\to} S(M, M)$ a dinatural transformation. Let us show that, π satisfies equation (3.2). Let $X \in \operatorname{vect}_{\Bbbk}$, $M \in \mathfrak{m}_A$ and let $(x_i), (f_i)$ be a pair of dual basis of X and X^*. The right hand side of equation (3.2) is

$$\begin{aligned} S(m_{X^*,X,M}, \mathrm{id}_M)\beta^X_{X\triangleright M,M}\pi_{X\triangleright M} &= \oplus_i S(m_{X^*,X,M}, \mathrm{id}_M)S(p^{X\triangleright M}_{f_i}, p^M_{x_i})\pi_{X\triangleright M} \\ &= \oplus_i S(m_{X^*,X,M}, \mathrm{id}_M)S(p^M_{x_i}p^{X\triangleright M}_{f_i}, \mathrm{id}_M)\pi_M \\ &= \oplus_i S(p^M_{x_i}p^{X\triangleright M}_{f_i}m_{X^*,X,M}, \mathrm{id}_M)\pi_M \\ &= S(\mathrm{ev}_X \otimes \mathrm{id}_M, \mathrm{id}_M)\pi_M. \end{aligned}$$

The second equality follows from the dinaturality of π, and the last equality follows from (3.7). ∎

Combining Proposition 3.12 and Proposition 3.14 we have the next result.

3.15. Corollary. *Assume $\mathcal{M}$ is a left $\mathcal{C}$-module category, $\mathcal{A}$ is a category, and $S : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ is a functor. Let $\lambda_M : \int_{M\in\mathcal{M}} S \dddot{\to} S(M, M)$ be the associated dinatural transformation of the usual end. There exists a monomorphism*

$$\varphi : \oint_{M\in\mathcal{M}} (S, \beta) \to \int_{M\in\mathcal{M}} S$$

such that

$$S(ev_X \triangleright \mathrm{id}_M, \mathrm{id}_M)\lambda_M\varphi = S(m_{X^*,X,M}, \mathrm{id}_M)\beta^X_{X\triangleright M,M}\lambda_{X\triangleright M}\varphi, \tag{3.9}$$

for any $X \in \mathcal{C}, M \in \mathcal{M}$.

■

Using the above Corollary, we can give another characterization of the module end. Essentially, this new description says that the module end is a subobject of the usual end, and it is universal among those subobjects with morphisms that satisfy (3.9).

3.16. Proposition. *Let (E, ψ) be a pair, where*

- *E is an object in $\mathcal{A}$;*
- *$\psi : E \to \int_{M\in\mathcal{M}} S$ is a morphism such that*

$$S(ev_X \triangleright \mathrm{id}_M, \mathrm{id}_M)\lambda_M\psi = S(m_{X^*,X,M}, \mathrm{id}_M)\beta^X_{X\triangleright M,M}\lambda_{X\triangleright M}\psi. \tag{3.10}$$

Then, there exists a unique map $h : E \to \oint_{M\in\mathcal{M}}(S, \beta)$ such that $\psi = \varphi \circ h$.

■

4. Applications to the theory of representations of tensor categories

Throughout this section $\mathcal{C}$ will denote a tensor category.

4.1. Natural module transformations as an end. For a pair of functors $F, G : \mathcal{A} \to \mathcal{B}$ between two abelian categories $\mathcal{A}, \mathcal{B}$, it is well known that, there is an isomorphism

$$\mathrm{Nat}\,(F,G) \simeq \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(F(A),G(A)).$$

In this Section, we generalize this result when F and G are $\mathcal{C}$-module functors.

Let $\mathcal{M},\mathcal{N}$ be $\mathcal{C}$-module categories, and $(F,c),(G,d) : \mathcal{M} \to \mathcal{N}$ be module functors. The functor

$$\mathrm{Hom}_{\mathcal{N}}(F(-),G(-)) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{vect}_{\Bbbk}$$

evaluated on functions $f : M \to M'$, $g : N \to N'$ in $\mathcal{M}$, is

$$\mathrm{Hom}_{\mathcal{N}}(F(f),G(g))(\alpha) = G(g)\circ\alpha\circ F(f),$$

for any $\alpha \in \mathrm{Hom}_{\mathcal{N}}(F(M),G(N))$.

4.2. Proposition. *For any pair of $\mathcal{C}$-module functors $(F,c),(G,d)$ the functor*

$$\mathrm{Hom}_{\mathcal{N}}(F(-),G(-)) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathrm{vect}_{\Bbbk}$$

has a pre-balancing given by

$$\beta^X_{M,N} : \mathrm{Hom}_{\mathcal{N}}(F(M),G(X\triangleright N)) \to \mathrm{Hom}_{\mathcal{N}}(F(X^*\triangleright M),G(N)) \tag{4.1}$$

$$\beta^X_{M,N}(\alpha) = (ev_X \triangleright \mathrm{id}_{G(N)})m^{-1}_{X^*,X,G(N)}(\mathrm{id}_{X^*}\triangleright d_{X,N}\alpha)c_{X^*,M},$$

for any $X \in \mathcal{C}, M, N \in \mathcal{M}$. There is an isomorphism

$$\mathrm{Nat}_m(F,G) \simeq \oint_{M\in\mathcal{M}} (\mathrm{Hom}_{\mathcal{N}}(F(-),G(-)),\beta).$$

Proof. It follows straightforward that, $\beta^X_{M,N}$ are natural isomorphisms with inverses given by

$$\left(\beta^X_{M,N}\right)^{-1}(\alpha) = d^{-1}_{X,N}(\mathrm{id}_X \triangleright \alpha c^{-1}_{X^*,M})m_{X,X^*,F(M)}(\mathrm{coev}_X \triangleright \mathrm{id}_{F(M)}). \tag{4.2}$$

For any $M \in \mathcal{M}$, define $\pi_M : \mathrm{Nat}_m(F,G) \to \mathrm{Hom}_{\mathcal{N}}(F(M),G(M))$ by $\pi_M(\alpha) = \alpha_M$. It follows easily that, π is a dinatural transformation. Let us show that π satisfies (3.2). Let $\alpha \in \mathrm{Nat}_m(F,G)$, $M \in \mathcal{M}$, then the left hand side of (3.2) evaluated in α is equal to

$$\alpha_M F(\mathrm{ev}_X \triangleright \mathrm{id}_M).$$

The right hand side of (3.2) evaluated in α is equal to

$$\begin{aligned}
&= (ev_X \triangleright \mathrm{id}_{G(M)}) m^{-1}_{X^*,X,G(M)} (\mathrm{id}_{X^*} \triangleright d_{X,M} \alpha_{X \triangleright M}) c_{X^*,X \triangleright M} F(m_{X^*,X,M}) \\
&= (ev_X \triangleright \mathrm{id}_{G(M)}) m^{-1}_{X^*,X,G(M)} (\mathrm{id}_{X^*} \triangleright (\mathrm{id}_X \triangleright \alpha_M) c_{X,M}) c_{X^*,X \triangleright M} F(m_{X^*,X,M}) \\
&= (ev_X \triangleright \mathrm{id}_{G(M)}) (\mathrm{id}_{X^* \otimes X} \triangleright \alpha_M) c_{X^* \otimes X,M} \\
&= \alpha_M (ev_X \triangleright \mathrm{id}_{F(M)}) c_{X^* \otimes X,M} = \alpha_M F(\mathrm{ev}_X \triangleright \mathrm{id}_M).
\end{aligned}$$

The second equality follows since α is a module natural transformation and satisfies (2.6), the third equality follows by the naturality of m and since c satisfies (2.3) and the last one follows from the naturality of c.

Let E be a vector space equipped with a dinatural transformation $\xi_M : E \to \mathrm{Hom}_{\mathcal{N}}(F(M), G(M))$ such that (3.2) is satisfied. Define $h : E \to \mathrm{Nat}_m(F,G)$ as follows. For any $v \in E$, $M \in \mathcal{M}$, $h(v)_M = \xi_M(v)$. It is clear, by definition, that $\pi \circ h = \xi$. We must prove that, for any $v \in E$, $h(v)$ is a natural module transformation, that is, we must show that equation (2.6) is fulfilled, which in this case is

$$d_{X,M} \xi_{X \triangleright M}(v) = (\mathrm{id}_X \triangleright \xi_M(v)) c_{X,M}, \tag{4.3}$$

for any $X \in \mathcal{C}$, $M \in \mathcal{M}$. Since ξ satisfies (3.2), then

$$\left(\beta^X_{X \triangleright M,M}\right)^{-1} \left(\xi_M(v) F(ev_X \triangleright \mathrm{id}_M) F(m^{-1}_{X^*,X,M})\right) = \xi_{X \triangleright M}(v),$$

for any $v \in E$. Using the definition of $\left(\beta^X_{X \triangleright M,M}\right)^{-1}$ given in (4.2), this equation is equivalent to

$$\begin{aligned}
&d_{X,M} \xi_{X \triangleright M}(v) = \left(\mathrm{id}_X \triangleright \xi_M(v) F((ev_X \triangleright \mathrm{id}_M) m^{-1}_{X^*,X,M}) c^{-1}_{X^*,X \triangleright M}\right) \\
&m_{X,X^*,F(X \triangleright M)} (\mathrm{coev}_X \triangleright \mathrm{id}_{F(X \triangleright M)}) \\
&= \left(\mathrm{id}_X \triangleright \xi_M(v) F(ev_X \triangleright \mathrm{id}_M)\right) (\mathrm{id}_X \triangleright c^{-1}_{X^* \otimes X,M} m^{-1}_{X^*,X,F(M)} \\
&(\mathrm{id}_{X^*} \triangleright c_{X,M})) m_{X,X^*,F(X \triangleright M)} (\mathrm{coev}_X \triangleright \mathrm{id}) \\
&= \left(\mathrm{id}_X \triangleright \xi_M(v)\right) (\mathrm{id}_X \otimes ev_X \triangleright \mathrm{id}_M) (\mathrm{id}_{X \otimes X^*} \triangleright c_{X,M}) (\mathrm{coev}_X \triangleright \mathrm{id}) \\
&= \left(\mathrm{id}_X \triangleright \xi_M(v)\right) c_{X,M}.
\end{aligned}$$

The second equality follows from (2.3), the third equality follows from the naturality of c, and the last one follows from the rigidity of $\mathcal{C}$. Hence, $\mathrm{Nat}_m(F,G)$ satisfies the required universal property. ∎

4.3. On the category of module functors. Assume that $\mathcal{C}, \mathcal{E}, \mathcal{D}$, are tensor categories. Assume also that $\mathcal{M}$ is a $(\mathcal{C}, \mathcal{E})$-bimodule category, and that $\mathcal{N}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category. Then, we can consider the functors

$$L = L_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N}), \tag{4.4}$$

$$L(M \boxtimes_{\mathcal{C}} N) = \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(-, M) \triangleright N,$$

$$\widetilde{L} = \widetilde{L}_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N}) \tag{4.5}$$
$$\widetilde{L}(M \boxtimes_{\mathcal{C}} N) = \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(M,-)^* \triangleright N,$$

Both functors are equivalences of $(\mathcal{E},\mathcal{D})$-bimodule categories. This fact was proven in [7, Thm. 3.20], see also [1]. The bimodule structure on the functor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is described in (2.11). We will give another proof of the fact that L and $\widetilde{L}$ are category equivalences, and we shall show an explicit description of a quasi-inverse using the module end of some functor in an analogous way as [13, Lemma 3.5].

4.4. Remark. *Our choice of the definition of the action on the right module category $\mathcal{M}^{\mathrm{op}}$, given in Section 2.2, using the right dual, is justified by the definition of the functors L and $\widetilde{L}$. If one changes the action on $\mathcal{M}^{\mathrm{op}}$ using left duals, then one has to modify the definition of the functors L and $\widetilde{L}$, so that they are well-defined.*

For later use, let us explain explicitly what it means that $\widetilde{L}$ is a bimodule functor. For any $Z \in \mathcal{D}$, $W \in \mathcal{E}$, $M \in \mathcal{M}, N \in \mathcal{N}$ we have natural isomorphisms

$$\widetilde{L}(W \triangleright M \boxtimes_{\mathcal{C}} N) \simeq \widetilde{L}(M \boxtimes_{\mathcal{C}} N) \circ (- \triangleleft W), \tag{4.6}$$

$$\widetilde{L}(M \boxtimes_{\mathcal{C}} N \triangleleft Z) \simeq (- \triangleleft Z) \circ \widetilde{L}(M \boxtimes_{\mathcal{C}} N). \tag{4.7}$$

Assume that, $\mathcal{M},\mathcal{N}$ are exact indecomposable as left $\mathcal{C}$-module categories, then there exist algebras $A, B \in \mathcal{C}$ such that $\mathcal{M} \simeq \mathcal{C}_A$, $\mathcal{N} \simeq \mathcal{C}_B$ as module categories. Recall that, if $M \in \mathcal{C}_A$ then, by Lemma 1.2 (i), ${}^*M$ has structure of left A-module. Exactness of the module category $\mathcal{M}$ is needed to use Proposition 2.14 (iii).

4.5. Lemma. *Assume as above that $\mathcal{M} = \mathcal{C}_A$, $\mathcal{N} = \mathcal{C}_B$. Denote by $(\mathbb{S}_{\mathcal{M}}, \phi)$ a relative Serre functor associated to $\mathcal{M}$. Then, the following statements hold.*

(i) The functor $\widetilde{L}_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is equivalent to the composition of functors

$$(\mathcal{C}_A)^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{C}_B \xrightarrow{{}^*(-)\boxtimes \mathrm{Id}} {}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B \xrightarrow{\otimes} {}_A\mathcal{C}_B \xrightarrow{R} \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B).$$

Recall the definition of the functor R given in Proposition 2.14, $R : {}_A\mathcal{C}_B \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B)$, $R(V)(X) = X \otimes_A V$. In particular, it follows that $\widetilde{L}$ is a category equivalence.

(ii) For any $M \in \mathcal{M}$, $N \in \mathcal{N}$, there exists a natural isomorphism of module functors

$$\widetilde{L}_{\mathcal{M},\mathcal{N}}(M \boxtimes_{\mathcal{C}} N) \simeq L_{\mathcal{M},\mathcal{N}}(M \boxtimes_{\mathcal{C}} N) \circ \mathbb{S}_{\mathcal{M}}. \tag{4.8}$$

In particular L is also an equivalence of categories.

PROOF. Existence of the relative Serre functor $\mathbb{S}_{\mathcal{M}}$ is ensured by the fact that $\mathcal{M}$ is exact. Part (i) follows from the computation of the internal hom given in Proposition 2.7 (ii). Let us prove (ii). It follows from Lemma 2.6 that, functors

$$\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(M,-)^*,\ {}^*\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M):\mathcal{M}\to\mathcal{C},$$

are equivalent as $\mathcal{C}$-module functors. Also, it follows from Lemma 2.6 that, functors

$${}^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,\mathbb{S}_{\mathcal{M}}(-)),\ \underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{op}}}(\mathbb{S}_{\mathcal{M}}(-),\overline{M}):\mathcal{M}\to\mathcal{C}$$

are equivalent as $\mathcal{C}$-module functors. This implies that, $\widetilde{L}_{\mathcal{M},\mathcal{N}}(M\boxtimes_{\mathcal{C}}N)$ is equivalent to the $\mathcal{C}$-module functor

$${}^*\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M)\triangleright N:\mathcal{M}\to\mathcal{N},$$

and $L_{\mathcal{M},\mathcal{N}}(M\boxtimes_{\mathcal{C}}N)\circ\mathbb{S}_{\mathcal{M}}$ is equivalent to the $\mathcal{C}$-module functor

$${}^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,\mathbb{S}_{\mathcal{M}}(-))\triangleright N:\mathcal{M}\to\mathcal{N}.$$

The natural isomorphisms $\phi_{U,V}:\underline{\mathrm{Hom}}(U,V)^*\to\underline{\mathrm{Hom}}(V,\mathbb{S}_{\mathcal{M}}(U))$ induce an isomorphism of $\mathcal{C}$-module functors

$${}^{**}\phi_{-,M}\triangleright \mathrm{id}_N:{}^*\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M)\triangleright N\to{}^{**}\underline{\mathrm{Hom}}_{\mathcal{M}}(M,\mathbb{S}_{\mathcal{M}}(-))\triangleright N.$$

And this finishes the proof of the Lemma. ∎

In what follows, we shall give an explicit description of a quasi-inverse of the functor $\widetilde{L}$ using the module end. For any module functor $(F,c)\in\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B)$ we introduce some auxiliary functors $S_F, D_F, \mathcal{L}_F, \mathcal{R}_F$ that, later, we will compute its module end.

Define

$$\begin{aligned} S_F:(\mathcal{C}_A)^{\mathrm{op}}\times\mathcal{C}_A&\to(\mathcal{C}_A)^{\mathrm{op}}\boxtimes_{\mathcal{C}}\mathcal{C}_B,\\ S_F(M,N)&=\overline{M}\boxtimes_{\mathcal{C}}F(N),\end{aligned}\tag{4.9}$$

endowed with a pre-balancing

$$\beta^X_{M,N}:S_F(M,X\triangleright N)\to S_F(X^*\triangleright M,N)$$

$$\beta^X_{M,N}=b^{-1}_{M,X,N}(\mathrm{id}_M\boxtimes_{\mathcal{C}}c_{X,N}).$$

Also

$$\begin{aligned} D_F:(\mathcal{C}_A)^{\mathrm{op}}\times\mathcal{C}_A&\to(\mathcal{C}_B)^{\mathrm{op}}\boxtimes_{\mathcal{C}}\mathcal{C}_A,\\ D_F(M,N)&=\overline{F(M)}\boxtimes_{\mathcal{C}}N,\end{aligned}\tag{4.10}$$

endowed with a pre-balancing

$$\delta^X_{M,N}:D_F(M,X\triangleright N)\to D_F(X^*\triangleright M,N)$$

$$\delta^X_{M,N} = (c^{-1}_{X^*,M} \boxtimes_{\mathcal{C}} \mathrm{id}_N) b^{-1}_{M,X,N}.$$

Here $b_{M,X,N} : X^* \triangleright M \boxtimes_{\mathcal{C}} N \to M \boxtimes_{\mathcal{C}} X \triangleright N$ is the balancing associated to the Deligne tensor product $\boxtimes_{\mathcal{C}}$, see Section 2.12.

We also have functors

$$\begin{aligned} \mathcal{L}_F, \mathcal{R}_F &: (\mathcal{C}_A)^{\mathrm{op}} \times \mathcal{C}_A \to {}_A\mathcal{C}_B, \\ \mathcal{R}_F(\overline{M}, N) &= {}^*M \otimes F(N), \\ \mathcal{L}_F(\overline{M}, N) &= {}^*F(M) \otimes N, \end{aligned} \tag{4.11}$$

equipped with pre-balancing

$$\begin{aligned} \gamma^X_{M,N} &: \mathcal{R}_F(M, X \triangleright N) \to \mathcal{R}_F(X^* \triangleright M, N), \\ \gamma^X_{M,N} &= ((\phi^l_{X^*,M})^{-1} \otimes \mathrm{id}_{F(N)})(\mathrm{id}_{{}^*M} \otimes c_{X,N}), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \eta^X_{M,N} &: \mathcal{L}_F(M, X \triangleright N) \to \mathcal{L}_F(X^* \triangleright M, N), \\ \eta^X_{M,N} &= {}^*(c_{X^*,M})(\phi^l_{X^*,F(M)})^{-1} \otimes \mathrm{id}_N. \end{aligned} \tag{4.13}$$

Here we are omitting the isomorphisms $X \simeq {}^*(X^*)$, for any $X \in \mathcal{C}$, and isomorphisms ϕ^l are those presented in (1.1).

4.6. Lemma. *Let $A, B \in \mathcal{C}$ be algebras such that module categories $\mathcal{C}_A, \mathcal{C}_B$ are exact. Let $(F, c) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$ be a $\mathcal{C}$-module functor. The following statements hold.*

(i) There exists an equivalence of categories ${}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id} : \mathcal{C}_A^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{C}_B \to {}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B$ such that

$$({}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id}\,) \circ \boxtimes_{\mathcal{C}} \simeq \boxtimes_{\mathcal{C}} \circ ({}^*(-) \times \mathrm{Id}\,)$$

as $\mathcal{C}$-balanced functors.

(ii) If the module end $\oint_{M \in \mathcal{C}_A}(S_F, \beta)$ exists, then

$$\widehat{\otimes} \circ ({}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id}\,)\Big(\oint_{M \in \mathcal{C}_A} (S_F, \beta)\Big) \simeq \oint_{M \in \mathcal{C}_A} (\mathcal{R}_F, \gamma).$$

(iii) If the module end $\oint_{M \in \mathcal{C}_A}(D_F, \delta)$ exists, then

$$\widehat{\otimes} \circ ({}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id}\,)\Big(\oint_{M \in \mathcal{C}_A} (D_F, \delta)\Big) \simeq \oint_{M \in \mathcal{C}_A} (\mathcal{L}_F, \eta).$$

Here $\widehat{\otimes} : {}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B \to {}_A\mathcal{C}_B$ is the induced functor from the tensor product, that we have presented in Proposition 2.14 (ii).

PROOF. Exactness of module categories $\mathcal{C}_A, \mathcal{C}_B$ is needed to ensure existence of functors S_F, D_F, see [1, Thm. 3.3 (4)]. Part (i) follows since $\boxtimes_{\mathcal{C}} \circ ({}^*(-) \times \mathrm{Id}\,)$ is a $\mathcal{C}$-balanced functor. There are isomorphisms of $\mathcal{C}$-balanced functors

$$\begin{aligned}\widehat{\otimes} \circ ({}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id}\,) \circ S_F &\simeq \widehat{\otimes} \circ ({}^*(-) \boxtimes_{\mathcal{C}} \mathrm{Id}\,) \circ \boxtimes_{\mathcal{C}} \circ (\mathrm{Id} \times F) \\ &\simeq \widehat{\otimes} \circ \boxtimes_{\mathcal{C}} \circ ({}^*(-) \times F) \\ &\simeq \otimes \circ ({}^*(-) \times F) = \mathcal{R}_F.\end{aligned}$$

The first isomorphism follows by the definition of S_F, the second isomorphism follows from part (i), and the third isomorphism is the one presented in Proposition 2.14 (ii). Now, part (ii) follows by applying Proposition 3.3 (i). The proof of (iii) follows similarly. ∎

4.7. THEOREM. *Let $A, B \in \mathcal{C}$ be algebras such that $\mathcal{C}_A, \mathcal{C}_B$ are exact module categories. The functor*

$$\Upsilon : \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B) \to \mathcal{C}_A^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{C}_B,$$

given by

$$\Upsilon(F) = \oint_{M \in \mathcal{C}_A} (S_F, \beta) = \oint_{M \in \mathcal{C}_A} \overline{M} \boxtimes_{\mathcal{C}} F(M)$$

is well-defined and is a quasi-inverse of the functor $\widetilde{L}$.

PROOF. Recall the definition of the functor R given in Proposition 2.14, $R : {}_A\mathcal{C}_B \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$, $R(V)(X) = X \otimes_A V$. It follows from Lemma 4.5 that, the composition of functors

$$(\mathcal{C}_A)^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{C}_B \xrightarrow{{}^*(-)\boxtimes \mathrm{Id}} {}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B \xrightarrow{\otimes} {}_A\mathcal{C}_B \xrightarrow{R} \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$$

is isomorphic to $\widetilde{L}$. Thus, it is enough to show that, the functor

$$\Psi : \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B) \to {}_A\mathcal{C}_B,$$

given by

$$\Psi(F) = \oint_{M \in \mathcal{C}_A} (\mathcal{R}_F, \gamma) = \oint_{M \in \mathcal{C}_A} {}^*M \otimes F(M) \tag{4.14}$$

is well-defined and it is a quasi-inverse of R. Since we know that R is an equivalence, we denote by Ψ an adjoint equivalence to R. Take $F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$, and $V \in {}_A\mathcal{C}_B$, then

$$\begin{aligned}\mathrm{Hom}_{(A,B)}(V, \Psi(F)) &\simeq \mathrm{Nat}_m(R(V), F) \\ &\simeq \oint_{M \in \mathcal{C}_A} (\mathrm{Hom}_B(M \otimes_A V, F(M)), \beta) \\ &\simeq \oint_{M \in \mathcal{C}_A} (\mathrm{Hom}_{(A,B)}(V, {}^*M \otimes F(M)), \delta) \\ &\simeq \mathrm{Hom}_{(A,B)}(V, \oint_{M \in \mathcal{C}_A} {}^*M \otimes F(M))\end{aligned}$$

The second isomorphism follows from Proposition 4.2. Here, the isomorphism β is the one described in (4.1). The third isomorphism follows from Lemma 1.2 (ii); one can easily verify that if

$$\delta^X_{M,N} : \mathrm{Hom}_{(A,B)}(V, {}^*M \otimes F(X \otimes N)) \to \mathrm{Hom}_{(A,B)}(V, {}^*M \otimes X \otimes F(N))$$

is defined as $\delta^X_{M,N}(h) = (\mathrm{id}_{{}^*M} \otimes c_{X,N}) \circ h$, then the naturality of Φ implies that

$$\delta^X_{M,N}(\Phi(\alpha)) = \Phi(\beta^X_{M,N}(\alpha)),$$

for any $\alpha \in \mathrm{Hom}_B(M \otimes_A V, F(X \otimes N))$. Here

$$\Phi : \mathrm{Hom}_B(M \otimes_A V, F(X \otimes N)) \to \mathrm{Hom}_{(A,B)}(V, {}^*M \otimes F(X \otimes N))$$

is the natural isomorphism described in (1.9). Thus, the third isomorphism follows by applying Proposition 3.3 (i). The last isomorphism follows from Proposition 3.3 (ii). ∎

As an immediate consequence of the above Theorem, we have the following results.

4.8. Corollary. *Let $A \in \mathcal{C}$ be an algebra such that $\mathcal{C}_A$ is an exact module category. There is an isomorphism of A-bimodules*

$$A \simeq \oint_{M \in \mathcal{C}_A} {}^*M \otimes M.$$

∎

4.9. Corollary. *Let $\mathcal{M}$, $\mathcal{N}$ be exact indecomposable $\mathcal{C}$-module categories. If $U \in \mathcal{M}$, $V \in \mathcal{N}$ and $F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, there are isomorphisms*

$$\oint_{M \in \mathcal{M}} \widetilde{L}_{\mathcal{M},\mathcal{N}}(\overline{M} \boxtimes_{\mathcal{C}} F(M)) \simeq F, \tag{4.15}$$

$$\oint_{M \in \mathcal{M}} \overline{M} \boxtimes_{\mathcal{C}} \widetilde{L}_{\mathcal{M},\mathcal{N}}(\overline{U} \boxtimes_{\mathcal{C}} V)(M) \simeq \overline{U} \boxtimes_{\mathcal{C}} V. \tag{4.16}$$

∎

4.10. Remark. *If $\mathcal{C} = \mathrm{vect}_{\Bbbk}$, Corollary 4.8 reduces to [4, Corollary 2.9].*

In [4, Lemma 3.8] it was proven that, for a right exact functor $F : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}, \mathcal{N}$ are abelian categories, there is an isomorphism

$$\int_{N \in \mathcal{N}} \overline{F^{r.a.}(N)} \boxtimes N \simeq \int_{M \in \mathcal{M}} \overline{M} \boxtimes F(M).$$

The next result is a generalization of that result; essentially it says that, for a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{N}$, there is an isomorphism

$$\oint_{N \in \mathcal{N}} \overline{F^{r.a.}(N)} \boxtimes_{\mathcal{C}} N \simeq \oint_{M \in \mathcal{M}} \overline{M} \boxtimes_{\mathcal{C}} F(M).$$

The proof, however, is more complicated than the proof of [4, Lemma 3.8], since in module ends there is a new ingredient (the pre-balancing β) that has to be taken into account.

4.11. Proposition. *Let $\mathcal{M},\mathcal{N}$ be exact indecomposable left $\mathcal{C}$-module categories. Assume that, $(F,c)\in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a module functor with right adjoint $(F^{r.a.},d)\in \mathrm{Fun}_{\mathcal{C}}(\mathcal{N},\mathcal{M})$. Recall the functors D_F, S_F defined in (4.10), (4.9) with their pre-balancings δ,β. There is an isomorphism*

$$\oint_{N\in\mathcal{N}} (\overline{F^{r.a.}(N)}\boxtimes_{\mathcal{C}} N,\delta)\simeq \oint_{M\in\mathcal{M}} (\overline{M}\boxtimes_{\mathcal{C}} F(M),\beta). \tag{4.17}$$

Proof. Since $\mathcal{M},\mathcal{N}$ are exact indecomposable, we can assume that, there are algebras $A,B\in\mathcal{C}$ such that $\mathcal{M}=\mathcal{C}_A,\mathcal{N}=\mathcal{C}_B$. Using Lemma 4.6 (ii), (iii), it will be enough to prove that there are isomorphisms

$$\oint_{N\in\mathcal{C}_B} (\mathcal{L}_{F^{r.a.}},\eta)\simeq \oint_{M\in\mathcal{C}_A} (\mathcal{R}_F,\gamma),$$

as objects in ${}_A\mathcal{C}_B$. Here η,γ are defined in (4.13), (4.12). Since the functor $R:{}_A\mathcal{C}_B\to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B)$, $R(V)(X)=X\otimes_A V$ is an equivalence of categories, using Proposition 3.3 (iii), it will be enough to prove that, there is an isomorphism

$$\oint_{N\in\mathcal{C}_B} \big(R({}^*F^{r.a.}(N)\otimes N),R(\eta)\big)\simeq \oint_{M\in\mathcal{C}_A} \big(R({}^*M\otimes F(M)),R(\gamma)\big). \tag{4.18}$$

Since the functor R is a quasi-inverse of the functor $\Psi:\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A,\mathcal{C}_B)\to {}_A\mathcal{C}_B$, presented in (4.14), it follows that

$$\oint_{M\in\mathcal{C}_A} \big(R({}^*M\otimes F(M)),R(\gamma)\big)\simeq F,$$

and

$$\oint_{N\in\mathcal{C}_B} \big(R({}^*N\otimes N),R(\gamma)\big)\simeq \mathrm{Id}_{\mathcal{C}_B}.$$

Hence, to prove isomorphism (4.18) of functors, it is sufficient to prove that, there is an isomorphism

$$\oint_{N\in\mathcal{C}_B} \big(R({}^*F^{r.a.}(N)\otimes N),R(\eta)\big)(U)\simeq \oint_{M\in\mathcal{C}_B} \big(R({}^*M\otimes M),R(\gamma)\big)(F(U))$$

For any $U\in\mathcal{C}_A$. Applying Theorem 3.9, it will be enough to prove that, there is an isomorphism

$$\oint_{N\in\mathcal{C}_B} \big(U\otimes_A {}^*F^{r.a.}(N)\otimes N,\widehat{\eta}\big)\simeq \oint_{M\in\mathcal{C}_B} \big(F(U)\otimes_B {}^*M\otimes M),\widehat{\gamma}\big),$$

where

$$\widehat{\eta}^X_{M,N}=R(\eta)_U=\mathrm{id}_U\otimes_A {}^*(d_{X^*,M})(\phi^l_{X^*,F^{r.a.}(M)})^{-1}\otimes \mathrm{id}_N,$$
$$\widehat{\gamma}^X_{M,N}=R(\gamma)_{F(U)}=\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1}\otimes \mathrm{id}_N,$$

for any $X \in \mathcal{C}$, $M, N \in \mathcal{C}_B$. For this purpose, we shall construct natural isomorphisms

$$a_{U,M} : F(U) \otimes_B {}^*M \to U \otimes_A {}^*F^{r.a.}(M)$$

such that

$$\widehat{\eta}^X_{M,N}(a_{U,M} \otimes \mathrm{id}_{X \otimes N}) = (a_{U,X^* \otimes M} \otimes \mathrm{id}_N)\widehat{\gamma}^X_{M,N}. \tag{4.19}$$

It will follow then from Proposition 3.3 (i) the desired isomorphism between module ends, and this will finish the proof of the Proposition.

Recall the isomorphisms $\Phi^A_{M,X,N}, \Psi^A_{M,X,N}$ defined in (1.15), (1.16)

$$\begin{aligned} \Phi^A_{M,X,N} &: \mathrm{Hom}_A(M, X \otimes N) \to \mathrm{Hom}_{\mathcal{C}}(M \otimes_A {}^*N, X), \\ \Phi^A_{M,X,N}(\alpha)\pi^A_{M,{}^*N} &= (\mathrm{id}_X \otimes \mathrm{ev}_N)(\alpha \otimes \mathrm{id}_{{}^*N}), \end{aligned}$$

$$\begin{aligned} \Psi^A_{M,X,N} &: \mathrm{Hom}_{\mathcal{C}}(M \otimes_A {}^*N, X) \to \mathrm{Hom}_A(M, X \otimes N), \\ \Psi^A_{M,X,N}(\alpha) &= (\alpha \pi^A_{M,{}^*N} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \mathrm{coev}_N). \end{aligned}$$

We shall also denote natural isomorphisms

$$\omega_{M,N} : \mathrm{Hom}_B(F(M), N) \to \mathrm{Hom}_A(M, F^{r.a.}(N)),$$

comming from the adjunction $(F, F^{r.a.})$. Naturality of ω implies that for any morphism $f : N \to \widetilde{N}$ in $\mathcal{C}_B$, and any $\alpha \in \mathrm{Hom}_B(F(M), N)$ we have that

$$\omega_{M,\widetilde{N}}(f\,\alpha) = F^{r.a.}(f)\omega_{M,N}(\alpha). \tag{4.20}$$

This equation implies in particular that

$$\omega_{U,Y \otimes N}(\Psi^B_{F(U),Y,M}(\mathrm{id}\,)) = F^{r.a.}(\Psi^B_{F(U),Y,M}(\mathrm{id}\,))\omega_{U,F(U)}(\mathrm{id}\,). \tag{4.21}$$

Define isomorphisms

$$a_{U,M} : F(U) \otimes_B {}^*M \to U \otimes_A {}^*F^{r.a.}(M)$$

induced by the natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(F(U) \otimes_B {}^*M, Z) \xrightarrow{\Psi^B} \mathrm{Hom}_B(F(U), Z \otimes M) \xrightarrow{\omega} \mathrm{Hom}_A(U, F^{r.a.}(Z \otimes M))$$

$$\to \mathrm{Hom}_A(U, Z \otimes F^{r.a.}(M)) \xrightarrow{\Phi^A} \mathrm{Hom}_{\mathcal{C}}(U \otimes_A {}^*F^{r.a.}(M), Z),$$

for any $Z \in \mathcal{C}$. This means that

$$a^{-1}_{U,M} = \Phi^A_{U,Y,F^{r.a.}(M)}\big(d_{Y,M}\, \omega_{U,Y \otimes N}(\Psi^B_{F(U),Y,M}(\mathrm{id}\,))\big),$$

where $Y = F(U) \otimes_B {}^*M$. Using the definition of Φ^A one gets that

$$\begin{aligned} a^{-1}_{U,M}\pi^A_{U,{}^*F^{r.a.}(M)} = (\mathrm{id}_Y \otimes \mathrm{ev}_{F^{r.a.}(M)}) \\ \big(d_{Y,M}\, \omega_{U,Y \otimes M}(\Psi^B_{F(U),Y,M}(\mathrm{id}\,)) \otimes \mathrm{id}_{{}^*F^{r.a.}(M)}\big). \end{aligned} \tag{4.22}$$

Here we are again denoting $Y = F(U)\otimes_B{}^*M$. Equation (4.19) is equivalent to

$$\begin{aligned}(a^{-1}_{U,X^*\otimes M}\otimes \mathrm{id}_N)\widehat{\eta}^X_{M,N}(\pi^A_{U,{}^*F^{r.a.\cdot}(M)}\otimes \mathrm{id}_{X\otimes N}) = \\ = \widehat{\gamma}^X_{M,N}(a^{-1}_{U,M}\otimes \mathrm{id}_{X\otimes N})(\pi^A_{U,{}^*F^{r.a.\cdot}(M)}\otimes \mathrm{id}_{X\otimes N}),\end{aligned}$$

which in turn (forgetting the last id_N) is equivalent to

$$\begin{aligned}(a^{-1}_{U,X^*\otimes M}\pi^A_{U,{}^*F^{r.a.\cdot}(X^*\otimes M)})(\mathrm{id}_U\otimes^*(d_{X^*,M})(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1}) = \\ = (\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1})(a^{-1}_{U,M}\pi^A_{U,{}^*F^{r.a.\cdot}(M)}\otimes \mathrm{id}_X).\end{aligned} \tag{4.23}$$

Using (4.22), the right hand side of (4.23) is equal to

$$\begin{aligned}&= (\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1})(a^{-1}_{U,M}\pi^A_{U,{}^*F^{r.a.\cdot}(M)}\otimes \mathrm{id}_X)\\
&= (\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1})\big(\mathrm{id}_{F(U)\otimes_B{}^*M}\otimes \mathrm{ev}_{F^{r.a.\cdot}(M)}\otimes \mathrm{id}_X\big)\big(d_{F(U)\otimes_B{}^*M,M}\\
&\omega_{U,F(U)\otimes_B{}^*M\otimes M}(\Psi^B_{F(U),F(U)\otimes_B{}^*M,M}(\mathrm{id}))\otimes \mathrm{id}_{{}^*F^{r.a.\cdot}(M)\otimes X}\big)\\
&= (\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1})\big(\mathrm{id}_{F(U)\otimes_B{}^*M}\otimes \mathrm{ev}_{F^{r.a.\cdot}(M)}\otimes \mathrm{id}_X\big)\big(d_{F(U)\otimes_B{}^*M,M}\\
&F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(M),M}(\mathrm{id}))\,\omega_{U,F(U)}(\mathrm{id})\otimes \mathrm{id}_{{}^*F^{r.a.\cdot}(M)\otimes X}\big)\end{aligned}$$

The last equality follows from (4.21). It follows from (4.22), that the left hand side of (4.23) is equal to

$$\begin{aligned}&= \big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)}\otimes \mathrm{ev}_{F^{r.a.\cdot}(X^*\otimes M)}\big)\big(d_{F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}\\
&\omega_{U,F(U)\otimes_B{}^*(X^*\otimes M)\otimes X^*\otimes M}(\Psi^B_{F(U),F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}(\mathrm{id}))\otimes \mathrm{id}_{{}^*F^{r.a.\cdot}(X^*\otimes M)}\big)\\
&\big(\mathrm{id}_U\otimes^*(d_{X^*,M})(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1}\big)\\
&= \big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)}\otimes \mathrm{ev}_{F^{r.a.\cdot}(X^*\otimes M)}\big)\big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)\otimes F^{r.a}(X^*\otimes M)}\otimes\\
&\otimes^*(d_{X^*,M})(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1}\big)\\
&\big(d_{F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}\,F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}(\mathrm{id}))\otimes \mathrm{id}\big)\\
&\big(\omega_{U,F(U)}(\mathrm{id})\otimes \mathrm{id}\big)\\
&= \big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)}\otimes \mathrm{ev}_{X^*\otimes F^{r.a.\cdot}(M)}(d_{X^*,M}\otimes(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1})\big)\\
&\big(d_{F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}\,F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}(\mathrm{id})\otimes \mathrm{id}\big)\\
&\big(\omega_{U,F(U)}(\mathrm{id})\otimes \mathrm{id}\big)\\
&= \big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)}\otimes \mathrm{ev}_{X^*\otimes F^{r.a.\cdot}(M)}(\mathrm{id}\otimes(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1})\big)\\
&\big(d_{F(U)\otimes_B{}^*(X^*\otimes M)\otimes X^*,M}\otimes \mathrm{id}_{{}^*F^{r.a.\cdot}(M)\otimes X}\big)\\
&\big(F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}(\mathrm{id})\otimes \mathrm{id}\big)\big(\omega_{U,F(U)}(\mathrm{id})\otimes \mathrm{id}\big).\end{aligned}$$

The second equation follows from (4.21), the third equality follows from (1.2), the fourth equality follows from (2.3) for the module functor $(F^{r.a.\cdot}, d)$, which in this case implies that

$$(\mathrm{id}\otimes d_{X^*,M})d_{F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M} = d_{F(U)\otimes_B{}^*(X^*\otimes M)\otimes X^*,M}.$$

At last, using the definition of Ψ^B, (1.3) and the rigidity axioms one can see that

$$\begin{aligned}
&\big(\mathrm{id}_{F(U)\otimes_B{}^*(X^*\otimes M)}\otimes \mathrm{ev}_{X^*\otimes F^{r.a.\cdot}(M)}(\mathrm{id}\otimes(\phi^l_{X^*,F^{r.a.\cdot}(M)})^{-1})\big)\\
&\big(d_{F(U)\otimes_B{}^*(X^*\otimes M)\otimes X^*,M}\otimes\mathrm{id}\big)\big(F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(X^*\otimes M),X^*\otimes M}(\mathrm{id}))\otimes\mathrm{id}\big)\\
&(\omega_{U,F(U)}(\mathrm{id})\otimes\mathrm{id}_{{}^*F^{r.a}(M)\otimes X})\\
&=(\mathrm{id}_{F(U)}\otimes_B(\phi^l_{X^*,M})^{-1})\big(\mathrm{id}_{F(U)\otimes_B{}^*M}\otimes\mathrm{ev}_{F^{r.a.\cdot}(M)}\otimes\mathrm{id}_X\big)\big(d_{F(U)\otimes_B{}^*M,M}\\
&F^{r.a.\cdot}(\Psi^B_{F(U),F(U)\otimes_B{}^*(M),M}(\mathrm{id}))(\omega_{U,F(U)}(\mathrm{id})\otimes\mathrm{id}_{{}^*F^{r.a}(M)\otimes X}).
\end{aligned}$$

This implies (4.23), and finishes the proof of the Proposition. ∎

4.12. A formula for the relative Serre functor. Let $\mathcal{M}$, $\mathcal{N}$ be exact indecomposable left $\mathcal{C}$-module categories, and recall the functors

$$L = L_{\mathcal{M},\mathcal{N}} : \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N}),$$

$$\widetilde{L} = \widetilde{L}_{\mathcal{N},\mathcal{M}^{\mathrm{bop}}} : \mathcal{N}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M}^{\mathrm{bop}} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{N}, \mathcal{M}^{\mathrm{bop}})$$

described in (4.4) and (4.5). Note that subindices of $\widetilde{L}$ are different to those presented in (4.5).

4.13. Lemma. *Use the above notation. For any $M \in \mathcal{M}$, $N \in \mathcal{N}$ there exists an equivalence of module functors*

$$L_{\mathcal{M},\mathcal{N}}(\overline{M}\boxtimes_{\mathcal{C}} N)^{l.a.\cdot} \simeq \widetilde{L}_{\mathcal{N},\mathcal{M}^{\mathrm{bop}}}(\overline{N}\boxtimes_{\mathcal{C}}\overline{\overline{M}}) \tag{4.24}$$

Proof. If $\mathcal{B}$ is an exact indecomposable right $\mathcal{C}$-module category, define the auxiliary functors

$$H^{\mathcal{B}}_B : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}, \quad \mathcal{R}^{\mathcal{N}}_N : \mathcal{C} \to \mathcal{N},$$
$$H^{\mathcal{B}}_B = \underline{\mathrm{Hom}}_{\mathcal{B}}(-, B), \quad \mathcal{R}^{\mathcal{N}}_N = - \triangleright N,$$

for any $B \in \mathcal{B}$, $N \in \mathcal{N}$. A straightforward computation shows that

$$(H^{\mathcal{B}}_B)^{l.a.\cdot}(X) = B \triangleleft X^*, \quad (\mathcal{R}^{\mathcal{N}}_N)^{l.a.\cdot}(N') = {}^*\underline{\mathrm{Hom}}_{\mathcal{N}}(N', N)$$

for any $X \in \mathcal{C}$, $N' \in \mathcal{N}$. Since $L(\overline{M}\boxtimes_{\mathcal{C}} N) = \mathcal{R}^{\mathcal{N}}_N \circ H^{\mathcal{M}^{\mathrm{op}}}_{\overline{M}}$, then

$$\begin{aligned}
L(\overline{M}\boxtimes_{\mathcal{C}} N)^{l.a.\cdot} &\simeq (H^{\mathcal{M}^{\mathrm{op}}}_{\overline{M}})^{l.a.\cdot} \circ (\mathcal{R}^{\mathcal{N}}_N)^{l.a.\cdot}\\
&\simeq \overline{M} \triangleleft \underline{\mathrm{Hom}}_{\mathcal{N}}(-, N) = \underline{\mathrm{Hom}}_{\mathcal{N}}(-, N)^* \triangleright M\\
&\overset{(2.6)}{\simeq} \underline{\mathrm{Hom}}_{\mathcal{N}^{\mathrm{op}}}(-, \overline{N})^{***} \triangleright M = \widetilde{L}(\overline{N}\boxtimes_{\mathcal{C}}\overline{\overline{M}}).
\end{aligned}$$

In the second equivalence, we are using the canonical isomorphisms ${}^*X^* \simeq X$. ∎

The next result is a formula for the relative Serre functor similar to the formula for the Nakayama functor given in [5]. Let $\mathcal{M}$ be an exact indecomposable left $\mathcal{C}$-module category. Let us denote by $\blacktriangleleft\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}^{\mathrm{op}}$ the action of the opposite module category, that is, the one determined by

$$\overline{M} \blacktriangleleft X = \overline{X^* \triangleright M}, \tag{4.25}$$

for any $M \in \mathcal{M}$, $X \in \mathcal{C}$. For any $M \in \mathcal{M}$ the functor

$$T_M : (\mathcal{M}^{\mathrm{op}})^{\mathrm{op}} \times \mathcal{M}^{\mathrm{op}} \to \mathcal{M},$$

$$T_M(U,V) = \underline{\mathrm{Hom}}_{\mathcal{M}}(M,V)^* \triangleright U,$$

has a pre-balancing

$$\gamma^X_{U,V} : T_M(U \blacktriangleleft X, V) \to T_M(U, V \blacktriangleleft {}^*X),$$

given as the composition

$$T_M(U \blacktriangleleft X, V) = \underline{\mathrm{Hom}}_{\mathcal{M}}(M,V)^* \triangleright (X^* \triangleright U) \xrightarrow{m^{-1}} (\underline{\mathrm{Hom}}_{\mathcal{M}}(M,V)^* \otimes X^*) \triangleright U$$

$$\to (X \otimes \underline{\mathrm{Hom}}_{\mathcal{M}}(M,V))^* \triangleright U \to \underline{\mathrm{Hom}}_{\mathcal{M}}(M, X \triangleright V)^* \triangleright U = T_M(U, V \blacktriangleleft {}^*X).$$

Thus we can consider the coend

$$\oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (T_M, \gamma).$$

Since T can be thought of as a functor $T : (\mathcal{M}^{\mathrm{op}})^{\mathrm{op}} \times \mathcal{M}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})$, $T(U,V)(M) = T_M(U,V)$, then using the parameter theorem described in Section 3.8, we have a functor

$$M \mapsto \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (T_M, \gamma).$$

We shall denote this functor as

$$\oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (T_-, \gamma) = \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (\underline{\mathrm{Hom}}_{\mathcal{M}}(-,U)^* \triangleright U, \gamma).$$

It follows from Lemma 2.5 that, $\oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (T_-, \gamma) : \mathcal{M} \to \mathcal{M}^{\mathrm{bop}}$ is a $\mathcal{C}$-module functor.

4.14. Theorem. *Let $\mathcal{M}$ be an exact indecomposable left $\mathcal{C}$-module category. There exists an equivalence of $\mathcal{C}$-module functors*

$$\mathbb{S}_{\mathcal{M}} \simeq \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (\underline{\mathrm{Hom}}_{\mathcal{M}}(-,U)^* \triangleright U, \gamma), \tag{4.26}$$

PROOF. Let $\mathcal{M},\mathcal{N}$ be a pair of exact indecomposable left $\mathcal{C}$-module categories. To prove the expression for the relative Serre functor, we will first compute a quasi-inverse of the functor $L = L_{\mathcal{M},\mathcal{N}}$ and then use equivalence (4.8).

Recall that $\blacktriangleleft\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}^{\mathrm{op}}$ is the action of the opposite module category. That is,

$$\overline{M} \blacktriangleleft X = \overline{X^* \triangleright M},$$

for any $M \in \mathcal{M}$, $X \in \mathcal{C}$. Let us denote by $D : (\mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N})^{\mathrm{op}} \to \mathcal{N}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M}^{\mathrm{bop}}$, the functor determined by $D(\overline{\overline{M} \boxtimes_{\mathcal{C}} N}) = \overline{N} \boxtimes_{\mathcal{C}} \overline{\overline{M}}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$. The functor D is an equivalence of categories.

Take $(F, c) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N})$. This means that, we have isomorphisms $c_{X,M} : F(X^{**} \triangleright M) \to X \triangleright F(M)$, for any $M \in \mathcal{M}, X \in \mathcal{C}$. Define

$$\Theta_F : (\mathcal{M}^{\mathrm{op}})^{\mathrm{op}} \times \mathcal{M}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N},$$

$$\Theta_F(\overline{\overline{U}}, \overline{V}) = \overline{V} \boxtimes_{\mathcal{C}} F(\overline{\overline{U}}).$$

The functor Θ_F has a pre-balancing

$$\nu^X_{U,V} : \Theta_F(\overline{\overline{U}} \blacktriangleleft X, \overline{V}) \to \Theta_F(\overline{\overline{U}}, \overline{V} \blacktriangleleft {}^*X),$$

$$\nu^X_{U,V} = b^{-1}_{V,{}^*X,F(U)}(\mathrm{id}_V \boxtimes_{\mathcal{C}} c_{{}^*X,U}).$$

Here $b_{V,X,U} : V \blacktriangleleft X \boxtimes_{\mathcal{C}} U \to V \boxtimes_{\mathcal{C}} X \triangleright U$ is the balancing of the $\mathcal{C}$-balanced functor $\boxtimes_{\mathcal{C}}$. Define $\chi : \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{bop}}, \mathcal{N}) \to \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N}$ the functor given by

$$\chi(F) = \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} (\Theta_F, \nu) = \oint^{\overline{U} \in \mathcal{M}^{\mathrm{op}}} \overline{U} \boxtimes_{\mathcal{C}} F(\overline{\overline{U}}).$$

The existence of these coends follows from the existence of the ends presented in Theorem 4.7 and the relation between ends and coends for left and right module categories given in Lemma 3.6. Let us prove that χ is a quasi-inverse of L. Since we already know that L is a category equivalence, it is enough to prove that

$$\chi(L(\overline{M} \boxtimes_{\mathcal{C}} N)) \simeq \overline{M} \boxtimes_{\mathcal{C}} N$$

for any $M \in \mathcal{M}, N \in \mathcal{N}$. Since D is a category equivalence, this is equivalent to prove that

$$D(\overline{\chi(L(\overline{M} \boxtimes_{\mathcal{C}} N))}) \simeq D(\overline{\overline{M} \boxtimes_{\mathcal{C}} N}) = \overline{N} \boxtimes_{\mathcal{C}} \overline{\overline{M}}. \tag{4.27}$$

for any $M \in \mathcal{M}, N \in \mathcal{N}$. The left hand side of (4.27) is equal to

$$\begin{aligned} D(\overline{\chi(L(\overline{M}\boxtimes_{\mathcal{C}}N))}) &= D\Big(\overline{\oint^{\overline{U}\in\mathcal{M}^{\mathrm{op}}} \overline{U}\boxtimes_{\mathcal{C}} L(\overline{M}\boxtimes_{\mathcal{C}}N)(\overline{U})}\Big) \\ &\simeq D\Big(\oint_{\overline{\overline{U}}\in\mathcal{M}^{\mathrm{bop}}} \overline{\overline{U}\boxtimes_{\mathcal{C}} L(\overline{M}\boxtimes_{\mathcal{C}}N)(\overline{\overline{U}})}\Big) \\ &\simeq \oint_{\overline{\overline{U}}\in\mathcal{M}^{\mathrm{bop}}} \overline{L(\overline{M}\boxtimes_{\mathcal{C}}N)(\overline{\overline{U}})}\boxtimes_{\mathcal{C}}\overline{\overline{U}} \\ &\simeq \oint_{V\in\mathcal{N}} \overline{V}\boxtimes_{\mathcal{C}} L(\overline{M}\boxtimes_{\mathcal{C}}N)^{l.a.}(V) \\ &\overset{(4.24)}{\simeq} \oint_{V\in\mathcal{N}} \overline{V}\boxtimes_{\mathcal{C}}\widetilde{L}(\overline{N}\boxtimes_{\mathcal{C}}\overline{\overline{M}})(V) \overset{(4.16)}{\simeq} \overline{N}\boxtimes_{\mathcal{C}}\overline{\overline{M}}. \end{aligned}$$

The first isomorphism follows from Lemma 3.7, the second one follows from Proposition 3.3 (iii), and the third isomorphism follows from Proposition 4.17.

Taking $\mathcal{N} = \mathcal{M}^{\mathrm{bop}}$ and using (4.8), it follows that

$$\widetilde{L}_{\mathcal{M},\mathcal{M}^{\mathrm{bop}}}(\chi(\mathrm{Id}\,)) \simeq L_{\mathcal{M},\mathcal{M}^{\mathrm{bop}}}(\chi(\mathrm{Id}\,)) \circ \mathbb{S}_{\mathcal{M}} \simeq \mathbb{S}_{\mathcal{M}},$$

and we obtain the desired description of the relative Serre functor. ∎

4.15. Remark. *If $\mathcal{C} = \mathrm{vect}_{\Bbbk}$ and $\mathcal{M}$ is a semisimple category, the (right) relatvie Serre functor coincides with the (right) Nakayama functor. In this case, formula (4.26) coincides with the formula for the (right) Nakayama functor presented in [4, Definition 3.14].*

4.16. Correspondence of module categories for Morita equivalent tensor categories. Assume that $\mathcal{C}, \mathcal{D}$ are Morita equivalent tensor categories. This means that, there is an invertible exact $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{B}$. We can assume that $\mathcal{D} = \mathrm{End}_{\mathcal{C}}(\mathcal{B})^{\mathrm{rev}}$, and the right action of $\mathcal{D}$ on $\mathcal{B}$ is given by evaluation

$$\triangleleft : \mathcal{B}\times\mathcal{D} \to \mathcal{B},\ B \triangleleft F = F(B).$$

It was proven in [3, Theorem 3.31] that, the maps

$$\mathcal{M} \mapsto \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathcal{M}), \quad \mathcal{N} \mapsto \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N})$$

are bijections, one the inverse of the other, between equivalence classes of exact $\mathcal{C}$-module categories and exact $\mathcal{D}$-module categories. We shall give another proof of this fact by showing an explicit equivalence of $\mathcal{D}$-module categories

$$\mathcal{N} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N})),$$

for any exact indecomposable $\mathcal{D}$-module category $\mathcal{N}$.

For any $(H,d) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N}))$, define

$$S_H : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{N}, \quad S_H(\overline{B}, C) = H(C)(\overline{B}).$$

This functor comes with isomorphisms

$$\beta^X_{B,C} : S_H(B, X \triangleright C) \to S_H(X^* \triangleright B, C),$$

$$\beta^X_{B,C} = \big(d_{X,C}\big)_B,$$

for any $X \in \mathcal{C}$, $B, C \in \mathcal{B}$.

4.17. Lemma. *The functor S_H is a $\mathcal{C}$-balanced functor with balancing given by $b_{B,X,C} : S_H(X^* \triangleright B, C) \to S_H(B, X \triangleright C)$, $b_{B,X,C} = \big(d_{X,C}\big)^{-1}_B$. In particular, there exists a right exact functor $\widehat{S}_H : \mathcal{B}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{B} \to \mathcal{N}$ such that $\widehat{S}_H \circ \boxtimes_{\mathcal{C}} \simeq S_H$ as $\mathcal{C}$-balanced functors.*

Proof. Since (H,d) is a module functor, the natural isomorphism d satisfy (2.3). This axiom implies that b satisfy (2.15). ∎

We can consider the functor

$$\Psi : \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N})) \to \mathcal{N},$$

$$\Psi(H) = \oint_{B \in \mathcal{B}} (S_H, \beta) = \oint_{B \in \mathcal{B}} H(B)(\overline{B}).$$

4.18. Proposition. *The functor Ψ is well-defined.*

Proof. The existence of the module end $\Psi(H)$ follows from applying the functor $\widehat{S}_H$ to the module end $\oint_{B \in \mathcal{B}} \overline{B} \boxtimes_{\mathcal{C}} B$, whose existence follow from Proposition 4.7, and using Proposition 3.3 (iii). ∎

4.19. Theorem. *Let $\mathcal{C}$ be a tensor category and $\mathcal{B}$ an indecomposable exact left $\mathcal{C}$-module category. Consider the finite tensor category $\mathcal{D} = \mathrm{End}_{\mathcal{C}}(\mathcal{B})^{\mathrm{rev}}$, and the functor $\widetilde{L} = \widetilde{L}_{\mathcal{B},\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{B} \to \mathrm{End}_{\mathcal{C}}(\mathcal{B})$ introduced in Section 4.3. Let $\mathcal{N}$ be an exact indecomposable left $\mathcal{D}$-module category. Define*

$$\Phi : \mathcal{N} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N})),$$

$$\Phi(N)(B)(\overline{C}) = \widetilde{L}(\overline{C} \boxtimes_{\mathcal{C}} B) \triangleright N,$$

for any $B, C \in \mathcal{B}$, $N \in \mathcal{N}$. The functors Φ and Ψ are well-defined, and they establish an adjoint equivalence of left $\mathcal{D}$-module categories

$$\mathcal{N} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N})).$$

PROOF. Take $N \in \mathcal{N}, B \in \mathcal{B}$. It follows immediately that, $\Phi(N)$ is a $\mathcal{C}$-module functor. That Φ and $\Phi(N)(B)$ are $\mathcal{D}$-module functors follow from the bimodule structure of the functor $\widetilde{L}$ (4.6), (4.7). Let us show that the pair of functors Φ, Ψ is an adjoint equivalence. Take $H \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{B}, \mathrm{Fun}_{\mathcal{D}}(\mathcal{B}^{\mathrm{op}}, \mathcal{N}))$, $C_1, C_2 \in \mathcal{B}$, then

$$\begin{aligned}
\Phi(\Psi(H))(C_1)(\overline{C_2}) &= \widetilde{L}(\overline{C_2}\boxtimes_{\mathcal{C}} C_1) \triangleright \oint_{B\in\mathcal{B}} H(B)(\overline{B}) \\
&\simeq \oint_{B\in\mathcal{B}} H(B)(\widetilde{L}(\overline{C_2}\boxtimes_{\mathcal{C}} C_1)^*(B)) \\
&\simeq \oint_{B\in\mathcal{B}} \widehat{S}_H\big(\widetilde{L}(\overline{C_2}\boxtimes_{\mathcal{C}} C_1)^*(B)\boxtimes_{\mathcal{C}} B\big) \\
&\simeq \widehat{S}_H\big(\oint_{B\in\mathcal{B}} \widetilde{L}(\overline{C_2}\boxtimes_{\mathcal{C}} C_1)^*(B)\boxtimes_{\mathcal{C}} B\big) \\
&\simeq \widehat{S}_H\big(\oint_{B\in\mathcal{B}} \overline{B}\boxtimes_{\mathcal{C}} \widetilde{L}(\overline{C_2}\boxtimes_{\mathcal{C}} C_1)(B)\big) \\
&\simeq \widehat{S}_H(\overline{C_2}\boxtimes_{\mathcal{C}} C_1) \simeq H(C_1)(\overline{C_2}).
\end{aligned}$$

The first isomorphism follows since $H(B)$ is a $\mathcal{D}$-module functor, the second isomorphism follows from the definition of $\widehat{S}_H$ given in Lemma 4.17, the third one follows from Proposition 3.3 (iii), the fourth isomorphism follows from (4.17), and the fifth isomorphism follows from (4.16).

Now, let us take $N \in \mathcal{N}$, then

$$\begin{aligned}
\Psi(\Phi(N)) = \oint_{B\in\mathcal{B}} \Phi(N)(B)(\overline{B}) &= \oint_{B\in\mathcal{B}} \widetilde{L}(\overline{B}\boxtimes_{\mathcal{C}} B) \triangleright N \\
&\simeq \mathrm{Id} \triangleright N \simeq N.
\end{aligned}$$

The isomorphism follows from (4.15). One can verify, in the above proof of $\Phi(\Psi(H)) \simeq H$ and in the proof of $\Psi(\Phi(N)) \simeq N$, that each pre-balancing is used properly. ∎

4.20. THE DOUBLE DUAL TENSOR CATEGORY. Let $\mathcal{M}$ be an exact indecomposable left $\mathcal{C}$-module category. Then the *dual tensor category* $\mathcal{C}^*_{\mathcal{M}} = \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is again a finite tensor category [3]. The category $\mathcal{C}^*_{\mathcal{M}}$ acts on $\mathcal{M}$ by evaluation:

$$\begin{aligned}
\mathcal{C}^*_{\mathcal{M}} \times \mathcal{M} &\to \mathcal{M}, \\
(F, M) &\mapsto F(M).
\end{aligned}$$

The category $\mathcal{M}$ is exact indecomposable over $\mathcal{C}^*_{\mathcal{M}}$, see [3, Lemma 3.25]. Whence, we can consider the tensor category $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} = \mathrm{End}_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M})$. There is a canonical tensor functor

$$\begin{aligned}
can &: \mathcal{C} \to (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}, \\
can(X)(M) &= X \triangleright M,
\end{aligned}$$

for any $X \in \mathcal{C}$, $M \in \mathcal{M}$. One can see that $can(X)$ is a $\mathcal{C}^*_{\mathcal{M}}$-module functor. It was proven in [3, Theorem 3.27] that the functor *can* is an equivalence of categories. We shall give an expression of a quasi-inverse of this functor.

Take $(G,d) \in (\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$. This means that we have natural isomorphisms

$$d_{F,M} : G(F(M)) \to F(G(M)),$$

for any $F \in \mathcal{C}_{\mathcal{M}}^*$, $M \in \mathcal{M}$. Let us denote

$$\mathcal{H}_{(G,d)} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C},$$

$$\mathcal{H}_{(G,d)}(M,N) = \underline{\mathrm{Hom}}(M, G(N)).$$

The functor $\mathcal{H}_{(G,d)}$ has a pre-balancing γ (seeing $\mathcal{M}$ as a left module category over $\mathcal{C}_{\mathcal{M}}^*$). For any $F \in \mathcal{C}_{\mathcal{M}}^*$ define

$$\gamma_{M,N}^F : \mathcal{H}_{(G,d)}(M, F(N)) \to \mathcal{H}_{(G,d)}(F^{l.a.}(M), N),$$

(Recall that $F^* = F^{l.a.}$) as the composition

$$\underline{\mathrm{Hom}}(M, G(F(N))) \xrightarrow{\underline{\mathrm{Hom}}(\mathrm{id}\,, d_{F,N})} \underline{\mathrm{Hom}}(M, F(G(N))) \to$$

$$\xrightarrow{(2.13)} \underline{\mathrm{Hom}}(F^{l.a.}(M), G(N)).$$

Explicitly, using (2.13), this isomorphism is

$$\gamma_{M,N}^F = \psi^Z_{F^{l.a.}(M),G(N)}\big(\Omega_{Z \triangleright M, G(N)}(\phi^Z_{M,F(G(N))}(\mathrm{id}_Z))b^{-1}_{Z,M}\big) \circ \underline{\mathrm{Hom}}(\mathrm{id}\,, d_{F,N})$$

where $Z = \underline{\mathrm{Hom}}(M, F(G(N)))$, and isomorphism b is the module structure of the functor $F^{l.a.}$. Recall the isomorphisms presented in (2.12)

$$\phi^X_{M,N} : \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M,N)) \to \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N),$$
$$\psi^X_{M,N} : \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \to \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M,N)),$$

associated to the pair of adjoint functors $(- \triangleright M, \underline{\mathrm{Hom}}(M,-))$.

4.21. Theorem. *Let $\mathcal{M}$ be an exact indecomposable left $\mathcal{C}$-module category. The functor $\Upsilon : (\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^* \to \mathcal{C}$ given by*

$$\Upsilon(G) = \oint_{M \in \mathcal{M}} (\underline{\mathrm{Hom}}(M, G(M)), \gamma)$$

is well-defined. The pair of functors (Υ, can) is an adjoint equivalence of categories.

PROOF. We shall prove that, there are natural isomorphisms

$$\mathrm{Nat}_m(can(X), G) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \Upsilon(G)).$$

Let us fix $X \in \mathcal{C}$ and $(G, d) \in (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}$. Using Proposition 4.2 we have that

$$\mathrm{Nat}_m(can(X), G) \simeq \oint_{M\in\mathcal{M}} \big(\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, G(M)), \beta\big). \tag{4.28}$$

Recall that $\mathcal{M}$ is thought of as a module category over $\mathcal{C}^*_{\mathcal{M}}$. According to (4.1), the pre-balancing β is, in this case,

$$\beta^F_{M,N} : \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, G(F(N))) \to \mathrm{Hom}_{\mathcal{M}}(X \triangleright F^{l.a.}(M), G(N)),$$

$$\beta^F_{M,N}(\alpha) = (\mathrm{ev}_F)_{G(N)} F^{l.a.}(d_{F,N}\alpha) b^{-1}_{X,M}.$$

Here $\mathrm{ev}_F : F^{l.a.} \circ F \to \mathrm{Id}$ is the evaluation of the adjoint pair $(F^{l.a.}, F)$. If we denote by $\Omega_{M,N} : \mathrm{Hom}_{\mathcal{M}}(M, F(N)) \to \mathrm{Hom}_{\mathcal{M}}(F^{l.a.}(M), N)$ natural isomorphisms, then $(\mathrm{ev}_F)_M = \Omega_{F(M),M}(\mathrm{id}_{F(M)})$, for any $M \in \mathcal{M}$.

Using Proposition 3.3 (ii) we can consider the module end

$$\oint_{M\in\mathcal{M}} \big(\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, G(M)), \widehat{\gamma}\big),$$

where, the pre-balancing in this case is

$$\widehat{\gamma}^F_{M,N} : \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, G(F(N))) \to \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(F^{l.a.}(M), G(M))),$$

$$\widehat{\gamma}^F_{M,N}(\alpha) = \gamma^F_{M,N} \circ \alpha.$$

4.22. CLAIM. *Isomorphisms*

$$\psi^X_{M,G(N)} : \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, G(N)) \to \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, G(N)))$$

commutes with the pre-balancings, that is

$$\widehat{\gamma}^F_{M,N} \circ \psi^X_{M,G(F(N))} = \psi^X_{F^{l.a.}(M),G(N)} \circ \beta^F_{M,N}. \tag{4.29}$$

As a consequence of this claim, using Proposition 3.3 (i), we get an isomorphism of module ends

$$\oint_{M\in\mathcal{M}} \big(\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, G(M)), \beta\big) \simeq \oint_{M\in\mathcal{M}} \big(\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, G(M)), \widehat{\gamma}\big)$$
$$\simeq \mathrm{Hom}_{\mathcal{C}}(X, \oint_{M\in\mathcal{M}} \big(\underline{\mathrm{Hom}}(M, G(M)), \gamma\big) = \mathrm{Hom}_{\mathcal{C}}(X, \Upsilon(G)).$$

The second isomorphism follows from Proposition 3.3 (ii). Combining this isomorphism with (4.28) we get the result.

It remains to prove the claim. Naturality of ψ, ϕ and b implies that

$$\psi^X_{M,N}(\alpha(h \triangleright \mathrm{id}_M)) = \psi^Y_{M,N}(\alpha) \circ h, \tag{4.30}$$

$$\underline{\mathrm{Hom}}(\mathrm{id}, f)\psi^X_{M,N}(\alpha) = \psi^X_{M,N'}(f \circ \alpha), \tag{4.31}$$

$$\phi^X_{M,N}(\alpha \circ h) = \phi^Y_{M,N}(\alpha)(h \triangleright \mathrm{id}_M), \tag{4.32}$$

$$b^{-1}_{Y,M}(h \triangleright \mathrm{id}_{F^{l.a.}(M)}) = F^{l.a.}(h \triangleright \mathrm{id}_M)b^{-1}_{X,M}, \tag{4.33}$$

for any morphism $h : X \to Y$ in $\mathcal{C}$ and any $f : N \to N'$, $M, N', N \in \mathcal{M}$.

Let $\alpha \in \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, G(F(N)))$, and $Z = \underline{\mathrm{Hom}}(M, F(G(N)))$, then the left hand side of (4.29) evaluated in α is equal to

$$\begin{aligned}
&\gamma^F_{M,N} \circ \psi^X_{M,G(F(N))}(\alpha) = \\
&= \psi^Z_{F^{l.a.}(M),G(N)}\big(\Omega_{Z \triangleright M,G(N)}(\phi^Z_{M,F(G(N))}(\mathrm{id}_Z))b^{-1}_{Z,M}\big)\underline{\mathrm{Hom}}(\mathrm{id}, d_{F,N}) \\
&\psi^X_{M,G(F(N))}(\alpha) \\
&= \psi^Z_{F^{l.a.}(M),G(N)}\big(\Omega_{Z \triangleright M,G(N)}(\phi^Z_{M,F(G(N))}(\mathrm{id}_Z))b^{-1}_{Z,M}\big)\psi^X_{M,F(G(N))}(d_{F,N}\alpha) \\
&= \psi^X_{F^{l.a.}(M),G(N)}\big(\Omega_{Z \triangleright M,G(N)}(\phi^Z_{M,F(G(N))}(\mathrm{id}_Z))b^{-1}_{Z,M} \\
&(\psi^X_{M,F(G(N))}(d_{F,N}\alpha) \triangleright \mathrm{id}_{F^{l.a.}(M)})\big) \\
&= \psi^X_{F^{l.a.}(M),G(N)}\big(\Omega_{Z \triangleright M,G(N)}(\phi^Z_{M,F(G(N))}(\mathrm{id}_Z)) \\
&F^{l.a.}(\psi^X_{M,F(G(N))}(d_{F,N}\alpha) \triangleright \mathrm{id}_M)b^{-1}_{X,M}\big) \\
&= \psi^X_{F^{l.a.}(M),G(N)}\big(\Omega_{F(G(N)),G(N)}(\mathrm{id})F^{l.a.}(h)b^{-1}_{X,M}\big)
\end{aligned}$$

The second equality follows from (4.31), the third equality follows from (4.30), the fourth follows from (4.33), the fifth equality follows from the naturality of Ω. In the last equality the map h is

$$h = \phi^Z_{M,F(G(N))}(\mathrm{id}_Z)(\psi^X_{M,F(G(N))}(d_{F,N}\alpha) \triangleright \mathrm{id}_M).$$

The right hand side of (4.29) evaluated in α is equal to

$$\psi^X_{F^{l.a.}(M),G(N)}\big(\Omega_{F(G(N)),G(N)}(\mathrm{id})F^{l.a.}(d_{F,N}\alpha)b^{-1}_{X,M}\big).$$

It remains to observe that $h = d_{F,N}\alpha$, which follows from (4.32). ∎

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