HOMOGENEOUS SPACES OF INFINITE-DIMENSIONAL LIE ALGEBRAS AND CHARACTERISTIC CLASSES OF FOLIATIONS

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In this article we introduce a new language to describe many problems of differential geometry: for example, problems connected with the theory of pseudogroups, Lie equations, foliations, characteristic classes, etc. This is the language of infinite-dimensional Lie algebras and their homogeneous spaces. It is closely connected with the general idea of formal differential geometry set forth by I. M. Gel'fand in his lecture at the International Congress of Mathematicians in Nice. In addition to a detailed account of the theory of homogeneous spaces of infinite-dimensional Lie algebras, this article contains applications of this theory to the characteristic classes of foliations. It also includes results on these questions from earlier papers by I. M. Gel'fand, D. A. Kazhdan, D. B. Fuks, and the authors.

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Introduction

This article gives an account of a new language convenient for many questions of differential geometry. This language expresses differential-geometric problems in terms of infinite-dimensional Lie algebras and their homogeneous spaces.

Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra,[1] S an infinite-dimensional manifold, and ϕ an action of $\mathfrak{g}$ on S, so that $\varphi:\mathfrak{g} \to \mathfrak{A}(S)$ is a homomorphism of $\mathfrak{g}$ into the Lie algebra $\mathfrak{A}(S)$ of vector fields on S. We say that ϕ defines on S the structure of a principal homogeneous space (or a principal $\mathfrak{g}$-space) if the induced mapping of $\mathfrak{g}$ into the tangent space of S is an isomorphism at every point.

At first glance it may seem that the theory of principal homogeneous spaces of infinite-dimensional Lie algebras does not give anything fundamentally new, but is a simple generalization to the infinite-dimensional case of the usual concept of principal homogeneous space. But this impression is incorrect. The theory of principal homogeneous spaces of infinite-dimensional algebras is richer than the corresponding theory of finite-dimensional algebras. By Lie's third theorem, for a finite-dimensional algebra $\mathfrak{g}$ there always exists a Lie group G whose Lie algebra is $\mathfrak{g}$. This means that the action of $\mathfrak{g}$ on S extends (perhaps only locally) to an action of the group G, so that the study of the homogeneous spaces of Lie algebras reduces essentially to that of the homogeneous spaces of the corresponding Lie groups. This is not true for infinite-dimensional Lie algebras. For them there are many interesting examples of a completely different nature. For example, let W_n be the Lie algebra of formal vector fields in n variables. It turns out that a principal W_n-space can be constructed on any n-dimensional manifold M. Namely, consider the space $S(M)$ of formal coordinate systems on M, that is, the space of infinite jets of diffeomorphisms of a neighborhood of the origin in $\mathbf{R}^n$ into M. The structure of a principal W_n-space can be introduced on $S(M)$ in a canonical way (see §4).

We shall show that many problems of differential geometry can be naturally formulated in terms of the principal homogeneous space $S(M)$. For example, the pseudogroup structure on a manifold M can be interpreted as the reduction of a principal fiber bundle π: $S(M) \to M$ to a subgroup (see §7).

The language we shall describe first appeared in a paper of Gel'fand and Kazhdan [1]. In essence, the same language was developed in an article of Guillemin and Sternberg [7], where it is used to study deformations of pseudogroup structures. However, infinite-dimensional Lie algebras and homogeneous spaces are not mentioned in this article, although their shadow is clearly visible. We draw the attention of the reader to the

1) The definitions of an infinite-dimensional Lie algebra and an infinite-dimensional manifold and a study of their properties are in §1.

connection between our theory and the general idea of formal differential geometry, which is explained at the end of Gel'fand's lecture [15].

This article was also written to show that it is possible to construct and study characteristic classes of foliations within the framework of the theory of homogeneous spaces of infinite-dimensional Lie algebras. In this theory, for example, Bott's theorem that the characteristic classes of a vector bundle $Q(F)$ normal to a leaf of a foliation are equal to zero (beginning with a certain dimension) becomes a trivial consequence of the theorem on the cohomology of W_n. Apart from known results we also obtain new results and examples concerning characteristic classes of foliations (see §10). In applications to the theory of characteristic classes, this article is a continuation of the paper [3] and a detailed account of its results. It should be noted that in the description of characteristic classes, unlike Haefliger [2] and Fuks [10], we do not use general categorical arguments, in particular, we avoid the concept of a classifying space. We refer the reader interested in these questions to the articles mentioned above.

To make our article independent of the papers of Gel'fand, Kazhdan, and Fuks, we include some of their results from [4] and [5].

We do not assume the reader to have any prior knowledge of the subject of this article, and we give detailed proofs of all theorems.

In conclusion we say a few words on the structure of this paper. It has three chapters.

In Chapter 1 we develop the general theory of homogeneous spaces of infinite-dimensional Lie algebras. In §1 we define and study infinite-dimensional manifolds, Lie groups, and Lie algebras. In §2 we introduce the concept of a principal homogeneous space and of a homogeneous space of an infinite-dimensional Lie algebra. In §3 we construct the characteristic homomorphism from the cohomology ring of the Lie algebra into the cohomology ring of any homogeneous space of this algebra. We need this homomorphism later to construct characteristic classes of foliations.

Chapter 2 is concerned with the study of homogeneous spaces of Lie algebras of a special type, namely the so-called transitive subalgebras W_n. In this chapter we show that the notions of foliation and pseudogroup structure can be introduced in terms of these homogeneous spaces. The construction and study of the principal W_n-space $S(M)$ occurs in §4, which occupies a critical place in the whole paper. The space $S(M)$ plays a central role in all further constructions. The definition of a transitive subalgebra L of W_n and several important examples are given in §5. In §6 we introduce the idea of an L-structure and L-foliation, which are useful for our purposes when we replace the usual concepts of pseudogroup and foliation. The connection between our concept of an L-structure and the classical concept of a transitive pseudogroup is described in §7.

Chapter 3 is devoted to the study of characteristic classes of L-foliations. These characteristic classes are constructed in §8. More precisely, by means of the characteristic homomorphism constructed in §3 we construct a homomorphism from the cohomology ring of L into the ring of characteristic classes in the category of L-foliations. In §9 an analogous construction is carried out for Γ-foliations. The aim of §9 is to show that characteristic classes can also be constructed without using L-foliations, using only Haefliger's definition of a Γ-foliation. In §10 we give examples of L-foliations, examine their characteristic classes, and make clear what our theory tells us about them. Finally, in §11 we show how to construct the classical characteristic classes in the category of principal G-bundles within the framework of our theory.

In the list of references we have attempted to include all papers on the subject of this article known to us.

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Chapter 1

GENERAL THEORY OF HOMOGENEOUS SPACES OF LIE ALGEBRAS

§1. Infinite-dimensional manifolds and Lie groups

A smooth infinite-dimensional manifold is usually understood to be a space obtained by pasting together open subsets of a "model" topological vector space (most frequently a Banach space) by means of isomorphisms satisfying a certain smoothness condition. In what follows we have to consider manifolds whose model space is $\mathbf{R}^\infty$, the projective limit of the finite-dimensional vector spaces $\mathbf{R}^k$.

In this section we construct on such a manifold the usual apparatus of differential geometry: vector fields, differential forms, etc. It will become clear that manifolds with the model space $\mathbf{R}^\infty$ have many of the properties of finite-dimensional manifolds. But we observe the following difference: For infinite-dimensional manifolds with the model space $\mathbf{R}^\infty$, both the theorem on the local integrability of vector fields and the implicit function theorem are false.

DEFINITION 1.1. The *space* $\mathbf{R}^\infty$ is the projective limit[1] of the system $0 \leftarrow \mathbf{R} \leftarrow \ldots \leftarrow \mathbf{R}^k \xleftarrow{\pi_k^{k+1}} \mathbf{R}^{k+1} \leftarrow \ldots$, where π_k^{k+1} is the projection defined by the formula $\pi_k^{k+1}(x_1, x_2, \ldots, x_k, x_{k+1}) = (x_1, \ldots, x_k)$.

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[1] We assume that if a space is the projective or inductive limit of topological spaces, then it is topologized by the corresponding projective or injective limit.

The natural projections $\mathbf{R}^l \to \mathbf{R}^k$ and $\mathbf{R}^\infty \to \mathbf{R}^k$ are denoted by π_k^l and π_k, respectively.

REMARK 1.1. Consider the projective system $E_0 \leftarrow \ldots \leftarrow E_k \overset{\pi_k^{k+1}}{\longleftarrow} E_{k+1}, \ldots$, where E_k is a finite-dimensional linear space over $\mathbf{R}$ with the usual topology, and the maps π_k^{k+1} are linear epimorphisms. Then the space $E = \text{proj lim } E_k$ is isomorphic as a topological vector space to one of the spaces $\mathbf{R}^n$ ($n = 0, 1, \ldots, \infty$). In particular, a closed linear subspace of $\mathbf{R}^\infty$ is isomorphic to one of the spaces $\mathbf{R}^n$.

To develop a theory of smooth infinite-dimensional manifolds we must first introduce the notions of a smooth function on an open set U of $\mathbf{R}^\infty$, of a smooth mapping $f\colon U \to \mathbf{R}^\infty$, and we must define what a vector field and a differential form on U are.

DEFINITION 1.2. A *smooth mapping on an open set* $U \subset \mathbf{R}^\infty$ *into a finite-dimensional space* E is a mapping $f\colon U \to E$ such that for every point $x \in U$ there exists a neighborhood V of x for which the mapping $f|_V$ can be represented in the form[1] $f_k \circ \pi_k$, where f_k is a C^∞-mapping $\pi_k(V) \to E$. We denote the space of smooth mappings $f\colon U \to E$ by $C^\infty(U, E)$.

Now let $E = \text{proj lim } E_k$ or $E = \text{ind lim } E_k$, where the spaces E_k are finite-dimensional. Then for any neighborhood $U \subset \mathbf{R}^\infty$ the spaces $C^\infty(U, E_k)$ form a projective (respectively, inductive) system.

DEFINITION 1.3. A mapping $f\colon U \to E$ is called *smooth* if for every point $x \in U$ there is a neighborhood V of x such that the mapping $f|_V$ belongs to proj lim $C^\infty(U, E_k)$ (respectively, ind lim $C^\infty(U, E_k)$). We denote the space of these mappings by $C^\infty(U, E)$.

REMARK 1.2. Let U and V be open subsets of $\mathbf{R}^\infty$, and let $f\colon U \to V$ be a mapping. If ϕ is a real-valued function on V, the formula $f^*(\phi) = \phi \circ f$ defines a mapping f^* from the space of real-valued functions on V into the space of real-valued functions on U. It is easy to check that the mapping $f\colon U \to V$ is smooth if and only if $f^*(C^\infty(V, \mathbf{R})) \subset C^\infty(U, \mathbf{R})$.

DEFINITION 1.4. A *smooth infinite-dimensional manifold* is a Hausdorff space S and a family of pairs (U_i, ϕ_i), where U_i is an open subset of S and ϕ_i is a homomorphism of U_i onto an open subset of $\mathbf{R}^\infty$, such that the following conditions are satisfied:

1) $\cup\, U_i = S$;

2) if $U_i \cap U_j \neq 0$, then the mapping $\phi_j \circ \phi_i^{-1}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ belongs to the space $C^\infty(\phi_i(U_i \cap U_j), \phi_j(U_i \cap U_j))$.

In order to extend to infinite-dimensional manifolds the concepts of a smooth vector field, a differential form, etc, we first define them locally;

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1) Here and in what follows we reserve the notation π_k for the projection restricted to an open subset of $\mathbf{R}^\infty$.

that is, we assume that $S = U$, where U is an open subset of $\mathbf{R}^\infty$.

DEFINITION 1.5. Let E be a finite-dimensional vector space. The *differential of a smooth mapping* f: $U \to E$ *at a point* $x \in U$ is the linear mapping $df(x)$: $\mathbf{R}^\infty \to E$ constructed in the following way. Let V be a neighborhood of x such that $f|_V = f_k \circ \pi_k$ (see Definition 1.2). We set $df(x) = df_k(\pi_k(x)) \circ \pi_k$, where df_k denotes the usual differential of the C^∞-mapping f_k.

DEFINITION 1.6. The *differential of a smooth mapping* f: $U \to V$ at a point $x \in U$ is the linear mapping $df(x)$: $\mathbf{R}^\infty \to \mathbf{R}^\infty$, that is, the projective limit of the differentials at x of the smooth mappings $\pi_k \circ f$: $U \to \mathbf{R}^k$.

DEFINITION 1.7. The *tangent bundle* $T(U)$ *of an open set* $U \subset \mathbf{R}^\infty$ is the triple $(U \times \mathbf{R}^\infty, U, p)$, where p is projection onto the first factor.

DEFINITION 1.8. A *vector field on an open set* U is a smooth section of the bundle $T(U)$, that is, a mapping ζ: $U \to U \times \mathbf{R}^\infty$ of the form $x \mapsto (x, \zeta(x))$, where $x \in U$ and $\zeta(x) \in C^\infty(U, \mathbf{R}^\infty)$.

(For convenience we identify the smooth mapping $\zeta(x)$ with the vector field itself.)

Let ζ be a vector field on U, and f a smooth function on U. We now define the derivative ζf of f along ζ.

Let $x \in U$. Then there is a neighborhood V of x in which $f|_V = f_k \circ \pi_k$ for some index k. Let $x_1, \ldots, x_k$ be coordinates in $\mathbf{R}^k$. Then $\pi_k \circ \zeta|_V = (\zeta_1, \ldots, \zeta_k)$, where $\zeta_1, \ldots, \zeta_k$ are smooth functions on V. For any point $y \in V$ we set

$$(\zeta f)(y) = \sum_{i=1}^{k} \zeta_i(y) \frac{\partial f_k}{\partial x_i}(\pi_k(y)).$$

Clearly, the value of ζf at y does not depend on the choice of V, and ζf belongs to $C^\infty(U, \mathbf{R})$.

The commutator of two vector fields ζ and η is defined in the usual way as the smooth mapping $[\zeta, \eta]$: $U \to \mathbf{R}^\infty$ satisfying the condition:

$$\pi_k \circ [\zeta, \eta] = \sum_{i=1}^{k} [\zeta, \eta]_i \frac{\partial}{\partial x_i},$$ where $[\zeta, \eta]_i = \zeta(\eta_i) - \eta(\zeta_i)$, and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ is a basis in $\mathbf{R}^k$. It is easy to verify that $[\zeta, \eta] f = \zeta\eta f - \eta\zeta f$ for every function f on U.

Now we consider the space of continuous linear functionals on $\mathbf{R}^\infty$, which we denote by $(\mathbf{R}^\infty)^*$. Since any continuous linear functional l on $\mathbf{R}^\infty$ can be represented in the form $\pi_k^* \circ l_k$, where l_k is a linear functional on $\mathbf{R}^k$, the space $(\mathbf{R}^\infty)^*$ can also be defined as the inductive limit of the system $0 \to (\mathbf{R})^* \to \ldots \to (\mathbf{R}^k)^* \to \ldots$, where $(\mathbf{R}^k)^*$ is the space dual to $\mathbf{R}^k$.

DEFINITION 1.9. The *cotangent bundle* $T^*(U)$ of U is the triple

$(U \times (\mathbf{R}^\infty)^*, U, p)$, where p is the projection onto the first factor. The i-th exterior power $\Lambda^i T^*(U)$ of the cotangent bundle is the triple $(U \times \Lambda^i(\mathbf{R}^\infty)^*, U, p)$, where Λ^i denotes the i-th exterior power.

DEFINITION 1.10. A *differential form of degree i on U* is a smooth section of the bundle $\Lambda^i T^*(U)$, that is, a map ω: $U \to U \times \Lambda^i(\mathbf{R}^\infty)^*$ of the form $x \mapsto (x, \omega(x))$, where $\omega(x)$ is a smooth map $U \to \Lambda^i(\mathbf{R}^\infty)^*$.

We denote the space of differential forms of degree i on U by $\Omega^i(U)$. A differential form with values in the space $E = \text{proj lim } E_k$, where the E_k are finite-dimensional, is an element of the space $\Omega^i(U, E) = \text{proj lim } \Omega^i(U) \otimes E_k$. It follows immediately from our definitions of smooth mappings that in some neighborhood V of any point $x \in U$ the differential form $\omega \in \Omega^i(U)$ can be represented as $\pi_k^*\omega_k$, where ω_k is a differential form defined on the neighborhood $\pi_k(V) \subset \mathbf{R}^k$. Therefore, the exterior differential in the space of the usual differential forms on $\pi_k(V)$ induces an exterior differential d: $\Omega^i(U) \to \Omega^{i+1}(U)$. The complex $0 \to \Omega^0(U) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^i(U) \xrightarrow{d} \ldots$ is called the *de Rham complex.*

LEt $\zeta_1, \ldots, \zeta_k$ be vector fields on U. We define the value $\omega(\zeta_1, \ldots, \zeta_i)$ of a differential form ω on $\zeta_1, \ldots, \zeta_i$ by the formula $\omega(\zeta_1, \ldots, \zeta_i)(x) = \omega(x)(\zeta_1(x), \ldots, \zeta_i(x))$, where $x \in U$, and $\omega(x)$ is regarded as an i-linear skew-symmetric functional on $\mathbf{R}^\infty$. The smoothness of $\omega(\zeta_1, \ldots, \zeta_i)$ follows immediately from that of ω.

PROPOSITION 1.1. (invariant formula for the differential d):

$$(1.1) \quad d\omega(\zeta_1, \ldots, \zeta_{i+1}) = \sum_{1 \leq s < t \leq i+1} (-1)^{s+t} \omega([\zeta_s, \zeta_t], \zeta_1, \ldots, \hat{\zeta}_s, \ldots, \hat{\zeta}_t, \ldots, \zeta_{i+1}) - \sum_k (-1)^k \zeta_k(\omega(\zeta_1, \ldots, \hat{\zeta}_k, \ldots, \zeta_{i+1})).$$

PROOF. We check (1.1) in a neighborhood V of x. Now ω coincides in V with the form $\pi_k^*\omega_k$. Therefore it can be represented as a linear combination of exterior products of 1-forms $f_\alpha dx_\alpha$, where f_α is a smooth function on V. Hence, it suffices to check the equality

$$d(f_\alpha\, dx_\alpha)(\xi, \eta) = -(f_\alpha\, dx_\alpha)([\zeta, \eta]) + \zeta((f_\alpha\, dx_\alpha)(\eta)) - \eta((f_\alpha\, dx_\alpha)(\zeta)),$$

where ζ and η are vector fields on V. Let

$$\pi_k \circ \zeta = \sum_{i=1}^{k} \zeta_i \frac{\partial}{\partial x_i}, \quad \pi_k \circ \eta = \sum_{i=1}^{k} \eta_i \frac{\partial}{\partial x_i}, \quad \pi_k \circ [\zeta, \eta] = \sum_{i=1}^{k} (\zeta(\eta_i) - \eta(\zeta_i)) \frac{\partial}{\partial x_i}.$$

Then, for large k,

$$(f_\alpha\, dx_\alpha)([\zeta, \eta]) = f_\alpha \left(\sum_{i=1}^{k} \zeta_i \frac{\partial \eta_\alpha}{\partial x_i} - \sum_{i=1}^{k} \eta_i \frac{\partial \zeta_\alpha}{\partial x_i} \right),$$

$$\zeta((f_\alpha\, dx_\alpha)(\eta)) = f_\alpha \left(\sum_{i=1}^{k} \zeta_i \frac{\partial \eta_\alpha}{\partial x_i} \right) + \left(\sum_{i=1}^{k} \zeta_i \frac{\partial f_\alpha}{\partial x_i} \right) \eta_\alpha,$$

$$\eta((f_\alpha\, dx_\alpha)(\zeta)) = f_\alpha \left(\sum_{i=1}^{k} \eta_i \frac{\partial \zeta_\alpha}{\partial x_i} \right) + \left(\sum_{i=1}^{k} \eta_i \frac{\partial f_\alpha}{\partial x_i} \right) \zeta_\alpha.$$

On the other hand,

$$d(f_\alpha\, dx_\alpha) = \sum_{i=1}^{k} \frac{\partial f_\alpha}{\partial x_i}\, dx_i \wedge dx_\alpha,$$

$$d(f_\alpha\, dx_\alpha)(\zeta, \eta) = \left(\sum_{i=1}^{k} \frac{\partial f_\alpha}{\partial x_i} \zeta_i \right) \eta_\alpha - \left(\sum_{i=1}^{k} \frac{\partial f_\alpha}{\partial x_i} \eta_i \right) \zeta_\alpha,$$

as required.

So far we have developed the local theory of infinite-dimensional manifolds and have extended the fundamental concepts of differential geometry to open subsets of $\mathbf{R}^\infty$.

Having the local theory, we can now extend this in the standard way (see [6], for example) to an arbitrary infinite-dimensional manifold. In particular, we can introduce the concepts of a tangent bundle $T(S)$, cotangent bundle $T^*(S)$, smooth vector field, de Rham complex $\Omega(S)$, etc. The cohomology ring $H^*(S)$ of the de Rham complex $\Omega(S)$ coincides, as in the finite-dimensional case, with the usual cohomology ring of the topological space S with coefficients in $\mathbf{R}$. To prove this it is enough to observe that the sequence

$$\mathbf{R} \to \tilde{\Omega}^0(S) \xrightarrow{d} \dots \xrightarrow{d} \tilde{\Omega}^k(S) \xrightarrow{d} \dots$$

of sheaves of germs of differential forms on S forms a resolution of the constant sheaf $\mathbf{R}$ on S.

Now we define an infinite-dimensional Lie group.

DEFINITION 1.11. An *infinite-dimensional Lie group* G is an infinite-dimensional manifold on which a group structure is defined in such a way that the operations of multiplication $G \times G \to G$ and inversion $G \to G$ are smooth mappings.

The Lie algebra of an infinite-dimensional Lie group can be defined in two ways:

1) the Lie algebra is the tangent space of the identity of G; or

2) the Lie algebra is the set of one-parameter subgroups of G.

In general, there is no exponential mapping in the infinite-dimensional case, and so the equivalence of these two definitions is not clear. However, in this paper we only deal with groups for which the exponential mapping exists, and then the two definitions of a Lie algebra are equivalent.

Infinite-dimensional manifolds appear in the following situation.

EXAMPLE 1.1. Suppose that the finite-dimensional C^∞-manifolds S^k and the smooth maps π^l_k: $S^l \to S^k$ (for $l > k$) form a projective system

$$S^0 \leftarrow S^1 \leftarrow \ldots \leftarrow S^k \xleftarrow{\pi_k^{k+1}} S^{k+1} \leftarrow \ldots$$

Assume that for some N and for all $k > N$ the mappings π^k_N: $S^k \to S^N$ are equipped with the structure of an affine bundle (that is, a bundle whose fiber is a vector space and whose structure group is the group of affine transformations of the vector space). We assume also that for $l > k > N$ the mappings π^l_k preserve the affine structure. Then the space S = proj lim S^k is a smooth finite- or infinite-dimensional manifold. Moreover, in this case the tangent bundle $T(S)$ can be identified in a natural way with proj lim $T(S^k)$, and the mappings π_k: $S \to S^k$ are smooth, locally trivial bundles.

We note that if the S^k form an arbitrary projective system, then proj lim S^k need not be a manifold in our sense of the word. For example, if all of the manifolds S^k are compact, then proj lim S^k is a compact space. But, clearly, a manifold with model space $\mathbf{R}^\infty$ cannot be compact.

§2. Homogeneous $\mathfrak{g}$-spaces

Gel'fand and Kazhdan [1] (see also [4]) have introduced for any nuclear Lie algebra $\mathfrak{g}$ the concept of a principal $\mathfrak{g}$-space S. We shall constantly use this idea, but in a slightly more special situation. Namely, we shall assume that the topological Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is, as a topological vector space, either $\mathbf{R}^k$ or $\mathbf{R}^\infty$. We assume that S is a smooth finite or infinite-dimensional manifold. Under these assumptions we can state the definition of Gel'fand and Kazhdan in the following way.

DEFINITION 2.1. A *principal homogeneous space* or simply a *principal $\mathfrak{g}$-space is a manifold* S together with a homomorphism ϕ from the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathfrak{A}(S)$ of vector fields on S, where ϕ satisfies the following conditions:

1) Let $T(S)$ denote the tangent bundle to S. Then for any point $s \in S$ the composite mapping φ_s: $\mathfrak{g} \to T_s(S)$ obtained by the restriction of the vector field $\phi(\zeta)$ for $\zeta \in \mathfrak{g}$ to the point s is an isomorphism of linear spaces.

2) The differential 1-form ω on S with values in $\mathfrak{g}$ defined by the formula $\omega(\zeta) = \phi_s^{-1}(\zeta)$ for $\zeta \in T_s(S)$ is a smooth form on S. (We call the form ω *canonical.*)

A principal $\mathfrak{g}$-space is, in particular, a manifold with absolute parallelism:

the tangent space of S at any point is canonically isomorphic to $\mathfrak{g}$.

PROPOSITION 2.1. (The Maurer-Cartan formula).

$$d\omega = -1/2[\omega, \omega]. \tag{2.1}$$

(We recall that, by definition, the differential form $[\omega, \omega]$ is defined by the formula $[\omega, \omega](\eta_1, \eta_2) = [\omega(\eta_1), \omega(\eta_2)] - [\omega(\eta_2), \omega(\eta_1)] = 2[\omega(\eta_1), \omega(\eta_2)]$, where $\eta_1, \eta_2 \in T_s(S)$.)

PROOF. Since the mapping ϕ_s is an isomorphism, it is sufficient to verify (2.1) for vector fields of the form $\phi(\zeta)$, where $\zeta \in \mathfrak{g}$. By the invariant formula (1.1) for the differential d we have

$$d\omega(\varphi(\zeta_1), \varphi(\zeta_2)) = -\omega([\varphi(\zeta_1), \varphi(\zeta_2)]) + \varphi(\zeta_1)\omega(\varphi(\zeta_2)) - \varphi(\zeta_2)\omega(\varphi(\zeta_1)).$$

Moreover, since ϕ is a homomorphism of Lie algebras, $[\phi(\zeta_1), \phi(\zeta_2)] = \phi([\zeta_1, \zeta_2])$. Also, $\omega(\phi(\zeta_i))$ is a constant function equal to ζ_i for $i = 1, 2$. Since $\omega = \phi^{-1}$, we obtain

$$d\omega(\varphi(\zeta_1), \varphi(\zeta_2)) = -[\zeta_1, \zeta_2] = -1/2[\omega, \omega](\varphi(\zeta_1), \varphi(\zeta_2)).$$

DEFINITION 2.2. Let $\mathfrak{h}$ be a closed subalgebra of $\mathfrak{g}$. A *homogeneous $\mathfrak{g}$-space with stationary subalgebra* $\mathfrak{h}$ is a triple (P, π, S), where

π: $P \xrightarrow{F} S$ is a bundle[1], P a principal $\mathfrak{g}$-space, and the fiber F a principal $\mathfrak{h}$-space. We assume here that the homomorphism $\varphi: \mathfrak{h} \to \mathfrak{A}(F)$ is contained in a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi_F} & \mathfrak{A}(P)|_F \\ \uparrow & & \uparrow \\ \mathfrak{h} & \xrightarrow{\varphi} & \mathfrak{A}(F), \end{array}$$

where ϕ_F is the restriction of $\varphi: \mathfrak{g} \to \mathfrak{A}(P)$ to the fiber F over a point of S, and the vertical arrows denote the natural inclusions. We also call S a *homogeneous $\mathfrak{g}$-space.*

§3. The characteristic homomorphism

In this section we construct a homomorphism from the standard complex of $\mathfrak{g}$ into the de Rham complex $\Omega(S)$ of a principal $\mathfrak{g}$-space S. We use only the fact that on S there is defined a $\mathfrak{g}$-valued differential form ω satisfying the Maurer-Cartan formula. First we recall several facts about the cohomology of topological Lie algebras.

DEFINITION 3.1. The *standard complex of cochains of a topological Lie algebra $\mathfrak{g}$ over* **R** is the complex $C(\mathfrak{g}) = \{C^q(\mathfrak{g}), d^q\}$, where $C^q(\mathfrak{g})$ is the space of continuous skew-symmetric q-linear real functionals on $\mathfrak{g}$, and the

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[1]) By bundle we mean a smooth, locally trivial bundle whose total space, base space, and fiber are smooth manifolds.

differential $d^q\colon C^q(\mathfrak{g}) \to C^{q+1}(\mathfrak{g})$ is defined by the formula

$$(dc)(\zeta_1, \ldots, \zeta_{q+1}) = \sum_{1 \leq s < t \leq q+1} (-1)^{s+t} c([\zeta_s, \zeta_t], \zeta_1, \ldots, \hat{\zeta}_s, \ldots, \hat{\zeta}_t, \ldots, \zeta_{q+1})$$
$$(\zeta_1, \ldots, \zeta_{q+1} \in \mathfrak{g}).$$

The homology of this standard complex is denoted $H^q(\mathfrak{g})$.

The standard complex $C(\mathfrak{g})$ is multiplicative: the product of cochains c_1 and c_2 is defined by the formula (where all the indices $i_1, \ldots, i_q$, $j_1, \ldots, j_r$ are distinct)

$$(c_1c_2)(\zeta_1, \ldots, \zeta_{q+r}) =$$
$$= \sum_{i_1 < \ldots i_q;\ j_1 < \ldots j_r} (-1)^{i_1 + \ldots + i_q - \frac{q(q+1)}{2}} c_1(\zeta_{i_1}, \ldots, \zeta_{i_q})\, c_2(\zeta_{j_1}, \ldots, \zeta_{j_r})$$
$$(c_1 \in C^q(\mathfrak{g}),\ c_2 \in C^r(\mathfrak{g})).$$

This product induces on the space $H^*(\mathfrak{g}) = \oplus_q H^q(\mathfrak{g})$ the structure of a graded ring. Apart from d, the following operations are defined on the standard complex $C(\mathfrak{g})$:

1. An operation of "inner product" $\iota(\zeta)$, where $\zeta \in \mathfrak{g}$, which is an antiderivation of degree (-1) on the graded ring $C(\mathfrak{g})$ and is defined on 1-cochains c by the formula $\iota(\zeta)c = c(\zeta)$.

2. An operation $\mathrm{ad}(\zeta)$, which is an antiderivation of degree 0 and is defined on 1-cochains c by the formula $(\mathrm{ad}(\zeta)c)(\eta) = -c([\zeta, \eta])$.

The operations $\iota(\zeta)$, $\mathrm{ad}(\zeta)$, and the differential d are connected by Weyl's formula:

$$\mathrm{ad}(\zeta) = d\iota(\zeta) + \iota(\zeta)\, d. \tag{3.1}$$

Moreover, for any $\zeta, \eta \in \mathfrak{g}$

$$\mathrm{ad}(\zeta)\iota(\eta) - \iota(\eta)\,\mathrm{ad}(\zeta) = \iota([\zeta, \eta]). \tag{3.2}$$

DEFINITION 3.2. Let $\mathfrak{h}$ be a closed subalgebra of $\mathfrak{g}$. A complex of relative cocycles $C(\mathfrak{g}, \mathfrak{h})$ is a subcomplex of $C(\mathfrak{g})$ consisting of the cocyles c for which

$$\iota(\zeta)c = \mathrm{ad}(\zeta)c = 0 \text{ for any } \zeta \in \mathfrak{h}. \tag{3.3}$$

The homology of the complex $C(\mathfrak{g}, \mathfrak{h})$ is denoted by $H^*(\mathfrak{g}, \mathfrak{h})$ and is called the *relative cohomology of the pair of algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$.

We note that (3.3) is equivalent to the equality

$$\iota(\zeta)c = \iota(\zeta)dc = 0. \tag{3.4}$$

It follows that $C(\mathfrak{g}, \mathfrak{h})$ is indeed a subcomplex.

REMARK 3.1. Suppose that the adjoint representation of a subalgebra $\mathfrak{h}$ in an algebra $\mathfrak{g}$ is extended to a representation of the group H (not necessarily connected) whose Lie algebra acts on $\mathfrak{h}$. Then there is a representation of H in the complex $C(\mathfrak{g})$. In the $C(\mathfrak{g}, \mathfrak{h})$ there is a distinguished subcomplex $C(\mathfrak{g}, H)$ of H-invariant cocycles, which is sometimes more convenient to use than $C(\mathfrak{g}, \mathfrak{h})$. We denote by $H^*(\mathfrak{g}, H)$ the homology of $C(\mathfrak{g}, H)$.

If S is a manifold, then for every vector field $\zeta \in \mathfrak{A}(S)$ the operations $\iota(\zeta)$ and $\mathrm{ad}(\zeta)$ are defined in the de Rham complex $\Omega(S)$. Namely, $\iota(\zeta)$ is the inner product of a vector field on a differential form, and the operation $\mathrm{ad}(\zeta)$ is the Lie derivative of a differential form along a vector field ζ, which is often denoted by $\mathscr{L}_\zeta$. The operations $\iota(\zeta)$ and $\mathscr{L}_\zeta$ are antiderivations of $\Omega(S)$ and satisfy Weyl's formulae (3.1) and (3.2). For any closed subalgebra $\mathfrak{B}(S) \subset \mathfrak{A}(S)$ the subcomplex of $\Omega(S)$ that consists of the differential forms satisfying (3.3) for every $\zeta \in \mathfrak{B}(S)$ is denoted by $\Omega(S, \mathfrak{B})$.

THEOREM 3.1. *Let S be a manifold, and ω a differential 1-form on S with values in $\mathfrak{g}$ satisfying the Maurer-Cartan formula $d\omega = -1/2\ [\omega, \omega]$. Then the mapping $\tilde{\Phi}: C(\mathfrak{g}) \to \Omega(S)$ defined by the formula*

$$\tilde{\Phi}(c)(\zeta_1, \ldots, \zeta_q) = c(\omega(\zeta_1), \ldots, \omega(\zeta_q)), \tag{3.5}$$

where $c \in C^q(\mathfrak{g})$ and $\tilde{\Phi}(c)(\zeta_1, \ldots, \zeta_q)$ is the value of $\tilde{\Phi}(c)$ on the vector fields $\zeta_1, \ldots, \zeta_q$, is a homomorphism of complexes.

COROLLARY 3.1. *For any principal $\mathfrak{g}$-space S with canonical form ω (3.5) defines a homomorphism of complexes $\tilde{\Phi}: C(\mathfrak{g}) \to \Omega(S)$.*

PROOF OF THEOREM 3.1. Since $\tilde{\Phi}$ is clearly a homomorphism of graded rings, it is sufficient to verify that $\tilde{\Phi}(dc) = d(\tilde{\Phi}(c))$ for $c \in C^1(\mathfrak{g})$. Let ζ and η be arbitrary vector fields on S. Then
$d(\tilde{\Phi}(c))(\zeta, \eta) = c(d\omega(\zeta, \eta))$. By the Maurer-Cartan formula,
$d\omega(\zeta, \eta) = -[\omega(\zeta), \omega(\eta)]$. Therefore
$d(\tilde{\Phi}(c))(\zeta, \eta) = -c([\omega(\zeta), \omega(\eta)]) = dc(\omega(\zeta), \omega(\eta)) = \tilde{\Phi}(dc)(\zeta, \eta)$.

We denote by Φ the homomorphism $H^*(\mathfrak{g}) \to H^*(S)$ induced by $\tilde{\Phi}$.

Now let S be a homogeneous $\mathfrak{g}$-space with stationary subalgebra $\mathfrak{h}$. By definition there is a bundle $\pi: P \to S$. Let $\mathscr{F}(P)$ denote the subalgebra of the Lie algebra $\mathfrak{A}(P)$ consisting of the vector fields tangent to the fibers of π.

PROPOSITION 3.1. *The image of $C(\mathfrak{g}, \mathfrak{h})$ under $\tilde{\Phi}$ is contained in $\Omega(P, \mathscr{F}(P))$.*

PROOF. Since for any vector field $\zeta \in \mathscr{F}(P)$ and any point $p \in P$ we can find an element $h \in \mathfrak{h}$ such that $\phi(h)|_p = \zeta|_p$, it is sufficient to check (3.4) for vector fields of the form $\phi(h)$. But for such vector fields this follows immediately from the definitions of $C(\mathfrak{g}, \mathfrak{h})$ and $\tilde{\Phi}$.

The projection $\pi: P \to S$ induces an embedding $\pi^*: \Omega(S) \to \Omega(P)$. We call a differential form on P *basic* if it lies in the image of π^*.

PROPOSITION 3.2. *Suppose that the fiber F of the bundle $\pi: P \to S$ is connected. Then the embedding $\pi^*: \Omega(S) \to \Omega(P)$ is an isomorphism of the complex $\Omega(S)$ onto the complex $\Omega(P, \mathscr{F}(P))$; in other words, the form $\alpha \in \Omega(P)$ is basic if and only if $\alpha \in \Omega(P, \mathscr{F}(P))$.*

PROOF. It is enough to prove that every point $p \in P$ has a neighborhood in which the form α is basic. We can therefore assume that $P = F \times U$, where F and U are open subsets of a linear space. Since α depends locally only on finitely many variables, we may assume that $U \subset \mathbf{R}^n$ and $F \subset \mathbf{R}^m$. Let $x_1, \ldots, x_n$ be coordinates in U and $y_1, \ldots, y_n$ coordinates in F. Any form $\alpha \in F(P)$ on $F \times U$ can be described as a sum of forms of the kind $f(y_1, \ldots, y_m, x_1, \ldots, x_n)dy_{i_1} \wedge \ldots \wedge dy_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l}$. A vector field $\zeta \in F(P)$, restricted to $F \times U$, can be written in the form $\Sigma\, g_i(y_1, \ldots, y_m, x_1, \ldots, x_n) \frac{\partial}{\partial y_i}$. Now let α satisfy (3.4) for some $\zeta \in F(P)$. The condition $\iota(\zeta)(\alpha) = 0$ means that $\alpha|_{F \times U}$ is a sum of forms of the kind $f dx_{j_1} \wedge \ldots \wedge dx_{j_l}$, and the condition $\iota(\zeta)(d\alpha) = 0$ means that f depends only on $x_1, \ldots, x_n$, as required.

The following theorem is a consequence of Propositions 3.1 and 3.2.

THEOREM 3.2. *The composition of the mapping* $\tilde{\Phi}$, *restricted to* $C(\mathfrak{g}, \mathfrak{h})$, *and the isomorphism* $(\pi^*)^{-1}$ *is a homomorphism of* $C(\mathfrak{g}, \mathfrak{h})$ *into* $\Omega(S)$.

We denote this homomorphism by $\tilde{\Phi}_{\mathfrak{h}}$, and its induced homomorphism $H^*(\mathfrak{g}, \mathfrak{h}) \to H^*(S)$ by $\Phi_{\mathfrak{h}}$.

REMARK 3.2. When F is a group (not necessarily connected), the analogous construction allows us to construct a mapping $\tilde{\Phi}_F: C(\mathfrak{g}, F) \to \Omega(S)$. More precisely, let $\pi: P \to S$ be a principal bundle with structure group F, and suppose that an action of F on $\mathfrak{g}$ is given. We assume that $\phi(a\zeta) = (R_a)_* \phi(\zeta)$ for any $a \in F$ and $\zeta \in \mathfrak{g}$, where R_a is the diffeomorphism of P corresponding to a. Propositions 3.1 and 3.2 automatically carry over to this case: the image of $C(\mathfrak{g}, F)$ under $\tilde{\Phi}$ consists of the basic forms, and the composition of $\tilde{\Phi}$ with the isomorphism $(\pi^*)^{-1}$ is a homomorphism of $C(\mathfrak{g}, F)$ $\Omega(S)$. The latter homomorphism is denoted by $\tilde{\Phi}_F$, and its induced homomorphism $H^*(\mathfrak{g}, F) \to H^*(S)$ by Φ_F.

Chapter 2

HOMOGENEOUS SPACES OF SUBALGEBRAS OF W_n AND THE THEORY OF FOLIATIONS

§4. The Lie algebra W_n. The basic example of a principal W_n-space.

In this section we define the Lie algebra W_n of formal vector fields in n variables and, following [1], we construct for each n-dimensional manifold M a principal W_n-space $S(M)$. The construction of the manifold $S(M)$ is of central importance in our paper.

We denote by $H(n)$ the space of formal power series in n variables $x_1, \ldots, x_n$. We introduce the filtration $H(n) = H_0(n) \supset \ldots \supset H_k(n) \supset \ldots$ where $H_k(n)$ consists of the formal power series whose formal derivatives of order less than k all vanish.

DEFINITION 4.1. A *formal vector field at the point* 0 *of* $\mathbf{R}^n$, or simply a *formal vector field in n variables*, is a linear combination $f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}$, where $f_1, \ldots, f_n \in H(n)$.

The set W_n of all formal vector fields in n variables provided with the obvious structure of a topological vector space and with the commutation operation given by the formula

$$\left[\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}\right] = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} f_i \frac{\partial g_k}{\partial x_i} - \sum_{j=1}^{n} g_j \frac{\partial f_k}{\partial x_j}\right) \frac{\partial}{\partial x_k},$$

is a topological Lie algebra.

In W_n there is a canonical filtration by subalgebras $L_i(W_n)$. Here $L_i(W_n)$, by definition, consists of the elements of the form $f_1 \frac{\partial}{\partial x_i} + \ldots + f_n \frac{\partial}{\partial x_n}$ where $f_1, \ldots, f_n \in H_{i+1}(n)$.

We denote by G_k^l the group of l-jets at the point $0 \in \mathbf{R}^n$ of diffeomorphisms of $\mathbf{R}^n$ whose k-jets coincide with the k-jet of the identity diffeomorphism. We set $G_k = \text{proj lim } G_k^l$. Then we have the following chain of inclusions: $G_0 \supset G_1 \supset \ldots \supset G_k \supset \ldots$.

There is a natural way of introducing on the groups G_k the structure of a Lie group. We construct an embedding α of G_0 in the linear space $(H_1(n))^n$ of all sets $(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in H_1(n)$. The embedding α is constructed in the following way.

For each diffeomorphism D of $\mathbf{R}^n$ that takes 0 to 0, and for each function $f \in C^\infty(\mathbf{R}^n)$, the formal Taylor series of D^*f at 0 depends only on the image of D in G_0. Hence for any $g \in G_0$ there is a well-defined mapping $g^*\colon C^\infty(\mathbf{R}^n) \to H(n)$. We set $\alpha(g) = (g^*(x_1), \ldots, g^*(x_n))$, where $x_1, \ldots, x_n$ are the coordinate functions on $\mathbf{R}^n$.

It is clear that a) $g^*(x_1) \in H_1(n)$; b) α is an open embedding of G_0 in $(H_1(n))^n$; c) the space $(H_1(n))^n$ is linearly isomorphic to $\mathbf{R}^\infty$; d) the embedding α defines on G_0 (and hence on all the groups G_k) the structure of an infinite-dimensional Lie algebra.

For $k > 0$, α determines a diffeomorphism of G_k onto the affine subspace $\tilde{H}_k$ of $(H_1(n))^n$ that consists of the sets $(f_1, \ldots, f_n)$ for which the formal power series $f_i - x_i \in H_{k+1}(n)$ $(i = 1, \ldots, n)$. Hence it follows that G_k for $k > 0$ is diffeomorphic to $\mathbf{R}^\infty$.

The Lie algebra of G_k can be identified in a natural way with $L_k(W_n)$. It follows from the formula $[L_i(W_n), L_j(W_n)] \subset L_{i+j}(W_n)$ that $L_k(W_n)$ for $k > 0$ is generalized nilpotent, that is, its descending central series contracts to zero. Hence for $k > 0$ the exponential mapping converges everywhere and defines a bijection $\exp\colon L_k(W_n) \to G_k$.

Now we construct for every n-dimensional C^∞-manifold M a principal W_n-space $S(M)$. We denote by $S^k(M)$ the manifold of k-jets at zero of regular mappings of some neighborhood of the origin of $\mathbf{R}^n$ into M. For $l > k$ there exists a natural projection $\pi_k^l : S^l(M) \to S^k(M)$, which assigns to each l-jet of mappings its k-jet. We consider the projective system $M = S^0(M) \leftarrow \ldots \leftarrow S^k(M) \leftarrow \ldots$ and set $S(M) = \text{proj lim } S^k(M)$. The space $S(M)$ is called *the space of formal coordinate systems on M* (or *the space of formal jets of local diffeomorphisms* $\mathbf{R}^n \to M$). It is easy to see that the triple $(S^l(M), S^k(M), \pi_k^l)$ is a principal fiber bundle with the structure group G_k^l. Consequently, $S(M)$ can be regarded as the total space of a principal fibering π_k: $S(M) \to S^k(M)$ whose structure group is G_k.

It is not difficult to see that the projective system $M \leftarrow \ldots \leftarrow S^k(M) \leftarrow \ldots$ satisfies the conditions of Example 1.1 with $N = 1$. Consequently, $S(M)$ is a smooth infinite-dimensional manifold, and the triple $(S(M), \pi_k, S^k(M))$ is a smooth locally-trivial fibration. We can immediately define an atlas on $S(M)$ in the following way.

Let U be some coordinate neighborhood of M, and $y_1, \ldots, y_n$ the coordinates on U. Now $S(U)$ is an open subset of $S(M)$. Every point $s \in S(U)$ defines a formal mapping $\mathbf{R}^n \to U$, in other words, in $(f_1, \ldots, f_n)$ the $f_i = s^*(y_i)$ $(i = 1, \ldots, n)$ are formal power series in n variables $x_1, \ldots, x_n$. Thus, we obtain an open embedding β_U of $S(U)$ in $(H(n))^n$, which is isomorphic to $\mathbf{R}^\infty$. It is easy to check that if $\{U_i\}$ is a covering of M by coordinate neighborhoods, then $\{S(U_i), \beta_{U_i}\}$ defines an atlas on $S(M)$. The action of G_0 on the space of the fibration π_0: $S(M) \to S^0(M) = M$ induces a homomorphism ϕ_0: $L_0(W_n) \to \mathfrak{A}(S(M))$. (This action is defined by the formula $R_g(s) = s \circ g^{-1}$, where $g \in G_0$ and $s \in S(M)$.)

The remaining part of this section is devoted mainly to a proof of the following result of great importance for our theory, which was first formulated by Gel'fand and Kazhdan (see [1], [4]).

THEOREM 4.1. *There exists a unique homomorphism* ϕ: $W_n \to \mathfrak{A}(S(M))$, *which:* 1) *extends the homomorphism* ϕ_0: $L_0(W_n) \to \mathfrak{A}(S(M))$; 2) *defines on* $S(M)$ *the structure of a principal* W_n*-space; and* 3) *is invariant under any diffeomorphism of* $S(M)$ *that induces a diffeomorphism of M.*

To construct this homomorphism ϕ we make use of the idea of lifting a vector field from M to $S(M)$.

DEFINITION – CONSTRUCTION 4.2. The k-jets of any local diffeomorphism τ of M induces a local diffeomorphism τ^k of $S^k(M)$. Any vector field ζ on M generates a one-parameter family of local diffeomorphisms $\tau(\zeta)_t$ of M. The family $\tau^k(\zeta)_t$ defines vector fields ζ^k on the $S^k(M)$, which are called *liftings of the vector field* ζ *to* $S^k(M)$.

It is clear from the construction of the lifting ζ^k of a vector field ζ that the value of ζ^k at any point $s \in S^k(M)$ depends only on the k-jet of ζ at $\pi_0^k(s)$. Therefore, the "lifting homomorphism" σ^k: $\mathfrak{A}(M) \to \mathfrak{A}(S^k(M))$, which takes ζ to ζ^k, induces a homomorphism σ_x^k: $J^k(M, x) \to T_s(S^k(M))$, where $J^k(M, x)$ denotes the space of k-jets of vector fields on M at x; $\pi_0^k(s) = x$. Clearly, this σ_x^k is a monomorphism. Moreover, the dimensions of the linear spaces $J^k(M, x)$ and $T_s(S^k(M))$ are equal. So we obtain:

PROPOSITION 4.1. *The homomorphism* σ_x^k *is an isomorphism.*

The tangent bundle $T(S(M))$ of $S(M)$ can naturally be identified with proj lim $T(S^k(M))$. We define a homomorphism $\sigma\colon \mathfrak{A}(M) \to \mathfrak{A}(S(M))$ by the formula $\sigma(\zeta) = \text{proj lim } \zeta^k$. We denote by $J^\infty(M, x)$ the space proj lim $J^k(M, x)$. From Proposition 4.1 we deduce the next result.

PROPOSITION 4.2. *For any point $s \in S(M)$ the homomorphism $\sigma_x\colon J^\infty(M, x) \to T_s(S(M))$, where $\pi_0(s) = x$ and $\sigma_x = \text{proj lim } \sigma_x^k$, is a linear isomorphism.*

CONSTRUCTION OF THE HOMOMORPHISM $\varphi\colon W_n \to \mathfrak{A}(S(M))$. Every diffeomorphism D from a neighborhood of zero in $\mathbf{R}^n$ to a neighborhood of x in M that takes 0 to x induces an isomorphism $J^\infty(M, x) \to J^\infty(\mathbf{R}^n, 0)$. Clearly, this isomorphism depends only on the formal jet at zero of D. Hence it follows that every point $s \in S(M)$ defines an isomorphism $\tilde{\alpha}_s\colon J^\infty(M, x) \to J^\infty(\mathbf{R}^n, 0)$, where $\pi_0(s) = x$. Since $J^\infty(\mathbf{R}^n, 0)$ is canonically isomorphic to W_n, we obtain an isomorphism $\alpha_s\colon J^\infty(M, x) \to W_n$. Let $\eta \in W_n$ and $s \in S(M)$. We have to construct a vector $\phi(\eta)|_s \in T_s(S(M))$. We set $\phi(\eta)|_s = -\sigma_x \circ \alpha_s^{-1}(\eta)$. Clearly, ϕ is continuous and extends ϕ_0; below we show that ϕ is a homomorphism of Lie algebras.

To prove that the mapping $\phi\colon W_n \to \mathfrak{A}(S(M))$ defines on $S(M)$ the structure of a principal W_n-space, we must construct on $S(M)$ a smooth differential 1-form ω with values in W_n, inverse to ϕ_s at every point s and satisfying the Maurer-Cartan formula $d\omega = -1/2[\omega, \omega]$. (Hence, in particular, ϕ is a homomorphism of Lie algebras.) We construct the required form ω in the following way. Let $\zeta_s \in T_s(S(M))$; we set $\omega(\zeta_s) = -\alpha_s \circ \sigma_x^{-1}(\zeta_s)$. Obviously, $\phi_s = \omega^{-1}$.

To prove the smoothness of ω we represent it as the limit of smooth differential forms on finite-dimensional manifolds $S^l(M)$.

Every point $s \in S^l(M)$ determines for $k < l$ a well-defined isomorphism $\tilde{\alpha}_s^k\colon J^k(M, x) \to J^k(\mathbf{R}^n, 0)$. Since $J^k(\mathbf{R}^n, 0)$ is canonically isomorphic to $W_n/L_k(W_n)$, we obtain an isomorphism $\alpha_s^k\colon J^k(M, x) \to W_n/L_k(W_n)$.

We define on $S^l(M)$ a form ω_k^l (for $l > k$) with values in the finite-dimensional linear space $W_n/L_k(W_n)$ by the formula $\omega_k^l(\zeta_s) = -\alpha_s^k \circ (\sigma_x^l)^{-1}(\zeta_s)$, where $\zeta_s \in T_s(S(M))$ and $\pi_0^l(s) = x$. The isomorphisms σ_x^l and α_s^k depend smoothly on s, and so ω_k^l is a smooth differential form. The smoothness of ω follows from $\omega = \text{proj lim } \omega_k$, where $\omega_k = \pi_l^* \omega_k^l$ does not depend on l.

To prove the Maurer-Cartan formula we require the following Lemma.

LEMMA 4.1. *Let τ be a diffeomorphism of M and τ^∞ the diffeomorphism of $S(M)$ induced by τ. Then $(\tau^\infty)^*\omega = \omega$.*

PROOF. Let ζ be a local vector field on N, defined in a neighborhood of y, and let $s\colon N \to M$ be a formal diffeomorphism of N into M that takes y to x. The formal diffeomorphism s defines a formal vector field $s_*\zeta$ at x. We consider the formal diffeomorphism $s\colon \mathbf{R}^n \to M$ that belongs to $S(M)$ and takes 0 to x. By definition, the value of ω on $\zeta_s = \sigma(\zeta)|_s$, where ζ is a local vector field on M, coincides with the formal vector field $s_*^{-1}(\zeta)$ at the point

$0 \in \mathbf{R}^n$. On the other hand, the value of $(\tau^\infty)^*\omega$ at the vector ζ_s is equal to $(\tau \circ s)_*^{-1}(\tau_*\zeta) = s_*^{-1}(\zeta)$.

COROLLARY 4.1. *For every vector field ζ on M we have $\mathcal{L}_{\sigma(\zeta)}\omega = 0$.*

We now prove the Maurer-Cartan formula. We must check that $d\omega(\zeta_s, \eta_s) = -[\omega(\zeta_s, \omega(\eta_s)]$ for any $\zeta_s, \eta_s \in T_s(S(M))$. We choose vector fields ζ and η on M such that $\sigma(\zeta)|_s = \zeta_s$ and $\sigma(\eta)|_s = \eta_s$. By the invariant formula for the differential,

$$d\omega(\sigma(\zeta), \sigma(\eta)) = \\ = -\omega([\sigma(\zeta), \sigma(\eta)]) + \mathcal{L}_{\sigma(\zeta)}(\omega(\sigma(\eta))) - \mathcal{L}_{\sigma(\eta)}(\omega(\sigma(\zeta))) + 2\omega([\sigma(\zeta), \sigma(\eta)]).$$

Substituting the vector fields $\sigma(\zeta)$ and $\sigma(\eta)$ in (3.2) and applying it to ω, we obtain

$$\mathcal{L}_{\sigma(\zeta)}(\omega(\sigma(\eta))) - \mathcal{L}_{\sigma(\eta)}(\omega(\sigma(\zeta))) = (\mathcal{L}_{\sigma(\zeta)}\omega)(\sigma(\eta)) - (\mathcal{L}_{\sigma(\eta)}\omega)(\sigma(\zeta)).$$

From Corollary 4.1 and the equality $[\sigma(\zeta), \sigma(\eta)] = \sigma([\zeta, \eta])$ we obtain $d\omega(\sigma(\zeta), \sigma(\eta)) = \omega(\sigma[\zeta, \eta])$. Finally,

$$d\omega(\zeta_s, \eta_s) = \omega(\sigma[\zeta, \eta])|_s = -s_*^{-1}([\zeta, \eta]) = -[s_*^{-1}\zeta, s_*^{-1}\eta] = -[\omega(\zeta_s), \omega(\eta_s)].$$

In order to complete the proof of Theorem 4.1 it remains to prove that the homomorphism ϕ is uniquely determined. Let φ': $W_n \to \mathfrak{A}(S(M))$ be another homomorphism extending φ_0: $L_0(W_n) \to \mathfrak{A}(S(M))$, defining at every point $s \in S(M)$ an isomorphism ϕ_s': $W_n \to T_s(S(M))$ and satisfying the condition $\tau_*^\infty(\phi'(\zeta)) = \phi'(\zeta)(\zeta \in W_n)$ for any diffeomorphism τ of M. We consider the mapping $A_s = (\phi_s')^{-1} \circ \phi_s$: $W_n \to W_n$ and claim that A_s is an automorphism of W_n.

Let $\zeta, \eta \in W_n$ and let $\hat{\zeta}, \hat{\eta}$ be vector fields on M such that $\sigma(\hat{\zeta})|_s = \phi_s(\zeta)$, $\sigma(\hat{\eta})|_s = \phi_s(\eta)$. Then $A_s([\zeta, \eta]) = \omega'([\sigma(\hat{\zeta}), \sigma(\hat{\eta})])$, where ω' is the canonical form of the principal W_n-space $(S(M), \phi')$. But by hypothesis ω' satisfies Lemma 4.1. It follows (see the proof of the Maurer-Cartan formula for ω) that $\omega'([\sigma(\hat{\zeta}), \sigma(\hat{\eta})]) = d\omega(\sigma(\hat{\zeta}), \sigma(\hat{\eta})) = [\omega'(\sigma(\hat{\zeta})), \omega'(\sigma(\hat{\eta}))]$. So we see that $A_s([\zeta, \eta]) = [\omega'(\phi_s(\zeta)), \omega'(\phi_s(\eta))] = [A_s(\zeta), A_s(\eta)]$.

The automorphism A_s: $W_n \to W_n$ is the identity on the subalgebra $L_0(W_n)$. It is easy to check that any such automorphism must be the identity on the whole of W_n. This means that $\phi_s = \phi_s'$ at every point s. This completes the proof of Theorem 4.1.

We now clarify the connection between the algebras $\sigma(\mathfrak{A}(M))$ and $\phi(W_n)$.

THEOREM 4.2. a) *A local vector field ζ on $S(M)$ belongs to the image of the homomorphism σ. $\mathfrak{A}(M) \to \mathfrak{A}(S(M))$ if and only if $[\zeta, \phi(W_n)] = 0$.*

b) *A local vector field ζ on $S(M)$ belongs to the image of the homomorphism ϕ: $W_n \to \mathfrak{A}(S(M))$ if and only if $[\zeta, \sigma(\mathfrak{A}(M))] = 0$.*

Theorem 4.2 is the infinitesimal analogue (and corollary) of the so-called first fundamental theorem.

THEOREM 4.3 (FIRST FUNDAMENTAL THEOREM). a) *Let ψ be any local diffeomorphism of $S(M)$. Then the equality $\psi = \tau^\infty$, where τ is some local diffeomorphism of M, is equivalent to the equality $\psi^*\omega = \omega$.*

b) *A local vector field ζ on $S(M)$ belongs to the image of the homomorphism φ: $W_n \to \mathfrak{A}(S(M))$ if and only if $\tau_*^\infty(\zeta) = \zeta$ for any local diffeomorphism τ of M.*

PROOF. a) follows easily from the first fundamental theorem as stated in [7] (see Theorem 4.1 there).

b) follows from the fact that a local diffeomorphism τ^∞ induced by a diffeomorphism τ acts transitively on $S(M)$.

In concluding this section we ask what the construction in §3 of the characteristic homomorphism gives for a principal $W_n$space $S(M)$.

EXAMPLE 4.1. On $S(M)$ there is a characteristic homomorphism Φ: $H^*(W_n) \to H^*(S(M))$. It turns out that this homomorphism is always zero.

EXAMPLE 4.2. Let M be a complex analytic manifold. We denote by $S^C(M)$ the space of formal analytic coordinate systems on M (that is, the space of formal jets at $0 \in \mathbf{C}^n$ of local analytic isomorphisms $\mathbf{C}^n \to M$). Let W_n^C denote the Lie algebra of formal holomorphic vector fields in n variables whose elements are linear combinations $f_1 \frac{\partial}{\partial z_1} + \ldots + f_n \frac{\partial}{\partial z_n}$, where $f_1, \ldots, f_n$ are formal power series with complex coefficients in $z_1, \ldots, z_n$.

As in the proof of Theorem 4.1, we can introduce on $S^C(M)$ the structure of a principal W_n^C-space. The characteristic homomorphism Φ: $H^*(W_n^C) \to H^*(S^C(M))$ is not always zero. This homomorphism will be described explicitly below (see §10, Theorem 10.9 and Remark 10.1).

§5. Transitive Lie algebras

The theory of homogeneous $\mathfrak{g}$-spaces that we have developed above will now be applied to the case when $\mathfrak{g}$ is a closed transitive subalgebra of W_n.

In this section we give the necessary definitions and some examples of transitive algebras. We need Examples 2 and 3 in §6 to define the notion of a foliation, Example 4 in §11 to construct the characteristic classes of principal G-bundles, Examples 5 – 9 to study characteristic classes of foliations of a special kind.

DEFINITION 5.1. A closed subalgebra L of W_n is called *transitive* if $\dim (L/L \cap L_0(W_n)) = n$.

We denote the subalgebra $L \cap L_k(W_n)$ by L_k.

For $k > 0$ we set $G_k(L) = \exp (L_k) \subset G_k$. Since $L_k(W_n)$ is generalized nilpotent, the Campbell–Hausdorff formula converges everywhere. It follows that $G_k(L)$ is a closed subgroup of the Lie group G_k. The mapping $\exp$: $L_k \to G_k(L)$ gives rise to a diffeomorphism from $G_k(L)$ to L_k.

We also wish to define the group $G_0(L)$. Since there are no natural reasons to assume that this group is connected, $G_0(L)$ is not uniquely determined. Henceforth, when speaking of transitive algebras, we always assume that together with an algebra L there is also given a closed subgroup $G_0(L)$ of G_0 whose Lie algebra is L_0.

Here are some important examples of transitive algebras:

1. The algebra W_n.

2. Consider the vector bundle p: $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, where p is the projection onto the first factor. Let $\mathcal{W}_{n,\ m}$ denote the Lie algebra of vector fields on $\mathbf{R}^n \times \mathbf{R}^m$ that preserve the fibers of p. The formal Taylor series at $0 \in \mathbf{R}^n \times \mathbf{R}^m$ of vector fields belonging to $\mathcal{W}_{n,\ m}$, form a transitive subalgebra $\mathcal{W}_{n,\ m}$ of W_{n+m}. It is easy to show that $W_{n, m}$ consists of linear combinations of the form

$$f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n} + g_1 \frac{\partial}{\partial y_1} + \ldots + g_m \frac{\partial}{\partial y_m},$$

where $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are coordinates in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively; $f_1, \ldots, f_n$ are arbitrary power series in $x_1, \ldots, x_n$; and $g_1, \ldots, g_m$ are arbitrary power series in $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$.

3. The construction in Example 2 can be generalized. Let L be any transitive subalgebra of W_n. We define a transitive subalgebra $W_n(L)$ of W_{n+m}. We represent $\mathbf{R}^{n+m}$ in the form $\mathbf{R}^n \times \mathbf{R}^m$. By definition, the elements of $W_m(L)$ are formal vector fields of the form $X + g_1 \frac{\partial}{\partial y_1} + \ldots + g_m \frac{\partial}{\partial y_m}$, where $X \in L$ and $g_1, \ldots, g_m$ are arbitrary power series in $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$. The elements of $W_m(L)$ for which $X = 0$ form an ideal V_{n+m} which does not depend on L.

The construction above has a group analogue. Let $G_0(L)$ be a subgroup of $G_0(W_n)$ whose Lie algebra is L_0. We define a subgroup $W_m(G_0(L))$ of $G_0(W_{n+m})$. The elements of $W_m(G_0(L))$ are formal coordinate changes of $\mathbf{R}^n \times \mathbf{R}^m$, defined by pairs (g, f), where $g \in G_0(L)$ and $f = (f_1, \ldots, f_n)$ is a set of power series in $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ taking 0 to 0 for which the matrix $\left(\frac{\partial f_i}{\partial y_j}(0)\right)$ is non-degenerate. The Lie algebra of $W_m(G_0(L))$ is, clearly, $L_0(W_m(L))$, and that of the normal subgroup $G_0(V_{n+m})$ is the ideal $V_{n, m}$ consisting of the pairs (id, f).

4. Let G be a connected finite-dimensional Lie group. We consider the principal G-bundle p: $\mathbf{R}^n \times G \to \mathbf{R}^n$ (p is the projection onto the first factor). We denote by $G(\mathcal{W}_n)$ the Lie algebra of vector fields on $\mathbf{R}^n \times G$ that preserve the fibers of the G-bundle p and are invariant under the action of G. To $G(\mathcal{W}_n)$ there corresponds the formal algebra $G(W_n)$ formed from the Taylor series at zero of vector fields belonging to $G(\mathcal{W}_n)$. The elements of $G(W_n)$ can be identified with expressions of the form $X + g$, where $X \in W_n$ and g is a formal power series in $x_1, \ldots, x_n$ with coefficients in the Lie algebra $\mathfrak{g}$ of G. The commutator $[X_1 + g_1, X_2 + g_2]$ is equal to $[X_1, X_2] + [g_1, g_2] + X_1 g_2 - X_2 g_1$, where $[X_1, X_2]$ is the commutator in W_n, $X_i g$ is the formal derivative series of g along $X_i (i = 1, 2)$, and $[g_1, g_2] = g_1 g_2 - g_2 g_1$.

5. The Lie algebra $W_n^{\mathbf{C}}$ of formal holomorphic vector fields in n variables (see Example 4.2). As a real algebra, $W_n^{\mathbf{C}}$ is a transitive subalgebra of W_{2n}.

6. Let $\alpha = dx_1 \wedge \ldots \wedge dx_n$ be the volume form in $\mathbf{R}^n$. The vector fields ζ in $\mathbf{R}^n$ for which $\mathcal{L}_\zeta\alpha = 0$ form a Lie algebra $\mathcal{A}_n$, which is called the algebra of non-divergent vector fields in $\mathbf{R}^n$. To this algebra there corresponds the algebra A_n of formal non-divergent vector fields in n variables, consisting of the formal Taylor expansions at zero of the vector fields $\zeta \in \mathcal{A}_n$.

7. We consider in $\mathbf{R}^{2n}$ the 2-form $\alpha = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$, where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are coordinates in $\mathbf{R}^{2n}$. The Hamiltonian Lie algebra $\mathcal{D}_{2n}$, by definition, consists of the vector fields ζ on $\mathbf{R}^n$ for which $\mathcal{L}_\zeta\alpha = 0$. We denote by D_{2n} the formal Hamiltonian Lie algebra corresponding to $\mathcal{D}_{2n}$.

8. We consider in $\mathbf{R}^{2n+1}$ the 1-form

$$\alpha = dt + \sum_{i=1}^{n} (x_i\, dy_i - y_i\, dx_i).$$

where $t, x_1, \ldots, x_n, y_1, \ldots, y_n$ are coordinates in $\mathbf{R}^{2n+1}$.
We define the algebra $\mathcal{K}_{2n+1}$ of contact vector fields on $\mathbf{R}^{2n+1}$ as the algebra of vector fields ζ for which $\mathcal{L}_\zeta\alpha = f\alpha$, where f is an arbitrary smooth function. To $\mathcal{K}_{2n+1}$ there corresponds the formal contact algebra K_{2n+1}.

9. Consider in $\mathbf{R}^{2n}$ the 1-form $\alpha = \sum_{i=1}^{n} e^{x_i}\, dy_i$,
where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are coordinates in $\mathbf{R}^{2n}$. The even-dimensional contact algebra $\mathcal{K}_{2n}$ consists, by definition, of the vector fields ζ for which $\mathcal{L}_\zeta\alpha = f\alpha$, where f is an arbitrary smooth function. To $\mathcal{K}_{2n}$ there corresponds the formal algebra K_{2n}.

REMARK 5.1. There is a considerable difference between the algebras K_{2n+1} and K_{2n}. The odd-dimensional algebra K_{2n+1} is primitive and simple (for the definition of primitive algebra, see, for example, [30]), but the even-dimensional algebra K_{2n} is not (for example, $K_2 = W_{1,1}$).

§6. L-structures and L-foliations

DEFINITION 6.1. Let L be a closed transitive subalgebra of W_n. A *regular L-structure*, or simply an *L-structure*, is a pair $(M, P(M))$, where M is an n-dimensional manifold and $P(M)$ is a smooth subbundle of the bundle $\pi: S(M) \to M$, that satisfies the conditions:

a) The bundle $\pi: P(M) \to M$ is the reduction of the principal G_0-bundle $\pi: S(M) \to M$ to $G_0(L)$.

b) $P(M)$ is a principal L-space with respect to the homomorphism $\varphi: L \to \mathfrak{A}(P(M))$, obtained by restriction of $\phi: W_n \to \mathfrak{A}(S(M))$ to L.

REMARK 6.1. If, for example, $L = W_n$ and $G_0(L)$ is the connected component of the identity of G_0, then the specification of an L-structure on M is equivalent to that of an orientation on M.

As we shall see later, the concept of an L-structure is closely connected with that of a pseudogroup structure on the manifold.

Now we shall give in these terms a definition of foliation with additional structure.

DEFINITION 6.2. Let L satisfy the conditions of Definition 6.1. An *L-foliation of codimension n on an $(n + m)$-dimensional manifold M* is an $W_m(L)$-structure $F = (M, P(M))$. (For convenience we assume, in addition, that $G_0(W_m(L)) = W_m(G_0(L))$.)

It is convenient to characterise an L-foliation $F = (M, P(M))$ not by the manifold $P(M)$, but by the factor manifold $P_F = P(M)/G_0(V_{n,m})$. The manifold P_F can be regarded as the total space of the principal bundle π_F: $P_F \to M$ with structure group $G_0(L) = W_m(G_0(L))/G_0(V_{n,m})$, which is the projective limit of the finite-dimensional principal $G_0^k(L)$-bundles π_F^k: $P_F^k \to M$. At the same time P_F serves as the base space of the bundle p_F: $P(M) \to P_F$ with fiber $G_0(V_{n,m})$. We identify P_F with a submanifold of the manifold of formal submersions $M \to \mathbf{R}^n$.

Let $s \in P_F$ and let $\pi_F(s) = x$. We choose an arbitrary point $s' \in p_F^{-1}(s)$. The formal submersion $\bar{s} = p \circ (s')^{-1}$ depends only on the choice of s. (We recall that we represent $\mathbf{R}^{n+m}$ in the form $\mathbf{R}^n \times \mathbf{R}^m$, and that p is the projection onto the first factor.) Conversely, suppose that the formal submersion $\bar{s}$ has the form $p \circ (s')^{-1}$, where $s' \in P(M)$. Then the point $s = p_F(s')$ depends only on $\bar{s}$.

Thus, by means of the mapping $s \mapsto \bar{s}$ the mapping P_F can be identified with a submanifold of the manifold of formal submersions $M \to \mathbf{R}^n$ (that is, the manifold of formal germs at $x \in M$ of submersions $\tilde{s}$: $U \to \mathbf{R}^n$, where $U \subset M$, $x \in U$, and $\tilde{s}(x) = 0$).

We now turn the class of L-foliations into a category.

DEFINITION 6.3. Let $F_1 = (M, P(M))$ and $F_2 = (N, P(N))$ be L-foliations. A morphism f: $F_1 \to F_2$ is a mapping of manifolds f: $M \to N$ such that for every formal submersion $\bar{s} \in P_{F_2}$ the formal mapping $\bar{s} \circ f$ is a submersion and belongs to P_{F_1}.

REMARK 6.2. Let $F = (N, P(N))$ be an L-foliation. We call a mapping f: $M \to N$ *transversal to the foliation F* if for every formal submersion $\bar{s} \in P_F$ the formal mapping $\bar{s} \circ f$ is also a submersion. In this case there is a uniquely determined structure of an L-foliation $f^*(F)$ on M such that f is a morphism $f^*(F) \to F$.

§7. The connection between L-structures and pseudogroup structures[1)]

We first recall several definitions (see [7]).

DEFINITION 7.1. A *transitive pseudogroup Γ on a manifold M* is a set

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1) The results of this section are not used in what follows.

of local diffeomorphisms ψ: $U_\psi \to V_\psi$ of M, (U_ψ and V_ψ being open subsets of M) that satisfies the following axioms:

1) *If* $\psi_1, \psi_2 \in \Gamma$ *and* $U_{\psi_1} = V_{\psi_2}$, *then* $\psi_1 \circ \psi_2 \in \Gamma$.

2) *If* $\psi \in \Gamma$, *then* $\psi^{-1} \in \Gamma$.

3) *If* $\psi \in \Gamma$ *and* $U \subset U_\psi$, *then* $\psi|_U \in \Gamma$.

4) *Let* ψ: $U_\psi \to V_\psi$ *be a local diffeomorphism such that for every* $x \in U_\psi$ *we can find an open set* U_x *for which* $\psi|_{U_x} \in \Gamma$. *Then* $\psi \in \Gamma$.

5) *The identity diffeomorphism id*: $M \to M$ *belongs to* Γ.

6) (Transitivity axiom). *For any* $x, y \in M$ *there exists a* $\psi \in \Gamma$ *such that* $\psi(x) = y$.

We always assume that a pseudogroup Γ also satisfies the following axioms:

AXIOM A. *We denote by* $G_0(\Gamma)$ *the group of formal jets at some point* $\Theta \in M$ *of diffeomorphisms* $\psi \in \Gamma$ *for which* $\psi(\Theta) = \Theta$. *Then* $G_0(\Gamma)$ *is a finite-dimensional or infinite-dimensional Lie group.*

AXIOM B. *We denote by* $S_\Theta(M)$ *the manifold of formal jets at* $\Theta \in M$ *of local diffeomorphisms* ψ: $U_\psi \to V_\psi$, *where* $U_\psi, V_\psi \subset M$ *and* $\Theta \in U_\psi$. *The subset* $P_\Theta(M) \subset S_\Theta(M)$, *that consists of formal jets of diffeomorphisms belonging to* Γ *is a smooth closed submanifold of the infinite-dimensional manifold* $S_\Theta(M)$.

AXIOM C. *A local diffeomorphism* ψ: $U_\psi \to V_\psi$ *belongs to* Γ *if and only if for every* $x \in U_\psi$ *we can find a diffeomorphism* $\psi_x \in \Gamma$ *such that the formal jets at* x *of the diffeomorphisms* ψ *and* ψ_x *coincide.*

DEFINITION 7.2. A local vector field on M is called a Γ-field if the one-parameter group of local diffeomorphisms generated by the field belongs to Γ.

The formal jets of Γ-fields at a fixed point $O \in M$ form an abstract[1] transitive Lie algebra $L_{abs}(\Gamma)$, which is isomorphic to a transitive subalgebra of W_n (where $n = \dim M$).

REMARK 7.1. When Γ acts on $\mathbf{R}^n$, we can choose the point 0 for Θ. Then the algebra $L_{abs}(\Gamma)$ can be identified canonically with the subalgebra $L(\Gamma)$ of W_n that consists of the formal Taylor expansions of Γ-fields at zero.

Now we show how the concepts of a pseudogroup and an L-structure are connected.

Let Γ be a transitive pseudogroup on a manifold M, and let $\Theta \in M$. We construct for Γ an L-structure on M, where L is isomorphic to $L_{abs}(\Gamma)$. For this purpose we choose a formal diffeomorphism s: $\mathbf{R}^n \to M$ that takes 0 to Θ. The formal diffeomorphism s induces an embedding $L_{abs}(\Gamma) \to W_n$ (expansion in a formal Taylor series of Γ-fields in the formal coordinate system s). The image of this embedding is an algebra $L \subset W_n$. The subset $P(M) \subset S(M)$ consisting of the formal diffeomorphisms $\psi \circ s$,

[1] A definition of an abstract transitive Lie algebra can be found, for example, in [30].

where $\psi \in \Gamma$, satisfies the conditions of Definition 6.1. Thus, the pair $(M, P(M))$ defines an L-structure on M. This L-structure is not uniquely determined, since $P(M)$ depends on the choice of s. Let s': $\mathbf{R}^n \to M$ be another formal diffeomorphism inducing an isomorphism $L_{abs}(\Gamma) \to L \subset W_n$. The manifold $P'(M)$ constructed from s' coincides with $P(M)$ constructed from s if and only if the formal diffeomorphism $s' \circ s^{-1}$ belongs to $G_0(\Gamma)$.

Conversely, let $(M, P(M))$ be an L-structure. We construct for this L-structure a pseudogroup Γ, namely we take a local diffeomorphism τ of M as belonging to Γ if and only if the diffeomorphism τ^∞ of $S(M)$ preserves $P(M)$. It is not difficult to verify that Γ satisfies axioms 1) – 5) and also the axioms A, B. C. But it is difficult to verify the transitivity axiom 6). We do not know whether every L-structure $(M, P(M))$ satisfies this axiom. To satisfy the transitivity axiom it is sufficient, for example, that the bundle π: $P(M) \to M$ satisfies the so-called Second Fundamental Theorem (see [7]).

THEOREM 7.1 (SECOND FUNDAMENTAL THEOREM). *Let $\tilde{\omega}$ be a local L-valued form on P(M) that satisfies the Maurer-Cartan formula $d\tilde{\omega} = -1/2\ [\tilde{\omega}, \tilde{\omega}]$. Then $\tilde{\omega} = \psi^*\omega$, where ω is the canonical L-valued form on P(M), and ψ is a local automorphism of the $G_0(L)$-bundle π: $P(M) \to M$.*

Chapter 3

CHARACTERISTIC CLASSES OF FOLIATIONS

§8. Characteristic classes in the category of L-foliations

In this section we show how to construct characteristic classes in the category of L-foliations by means of the characteristic homomorphism constructed in §3.

DEFINITION 8.1. Let C be an arbitrary category, and let T be a covariant (respectively, contravariant) functor from C to the category of sets. A *characteristic class in C with values in T* is a function χ that assigns to every object $c \in C$ an element $\chi(c) \in T(c)$ such that for every morphism α: $c \to c'$ we have $\chi(c') = T(\alpha)\chi(c)$ (respectively, $\chi(c) = T(\alpha)\chi(c')$).

We consider the category of L-foliations and the functor Ω_L, that assigns to every L-foliation F the space of L-valued differential forms on the manifold P_F.

We construct a characteristic class in the category of L-foliations with values in Ω_L. For this purpose we define on $P(M)$ an L-valued differential form ω_L by the formula $\omega_L = \omega / V_{n, m}$. (We recall that ω is the canonical $W_m(L)$-valued form on the principal $W_m(L)$-space $P(M)$.) From Proposition 3.1 we conclude that ω_L is basic for the bundle p_F: $P(M) \to P_F$. We denote by $\omega_L(F)$ the corresponding form on the base space P_F. Clearly,

$f^*\omega_L(F) = \omega_L(f^*F)$. So we obtain the following theorem.

THEOREM 8.1. *The function ω_L that takes the L-foliation F to the differential form $\omega_L(F)$ is a characteristic class in the category of L-foliations with values in Ω_L.*

Since ω satisfies the Maurer-Cartan formula, so does the form ω_L. Therefore (see Proposition 3.2) there are defined a homomorphism of complexes Φ: $C(L) \to \Omega(P_F)$ and a homomorphism Φ: $H^*(L) \to H^*(P_F)$. We denote by Ω the functor that assigns to a foliation F the de Rham complex of P_F. Since $\tilde{\Phi}$ commutes with the morphisms of the foliation, we have the following theorem.

THEOREM 8.2. a) $\tilde{\Phi}$ *is a homomorphism of the standard complex of cocycles of the topological algebra L into the ring of characteristic classes in the category of L-foliations with values in Ω.*

b) Φ *is a homomorphism of the cohomology ring of L into the ring of characteristic classes with values in the functor that assigns to an L-foliation F the cohomology ring of P_F.*

We investigate now how to compute the cohomology of P_F. To do this we consider the bundle $\pi^1: P_F \to P_F^1$. (We recall that P_F^1 is the manifold of 1-jets of formal submersions $s \in P_F$; in particular, P_F^1 is finite-dimensional.) The fiber of this bundle is diffeomorphic to the group $G_1(L)$, that is, to a linear space. Therefore π_1 is a homotopy equivalence, and π_1^*: $H^*(P_F^1) \to H^*(P_F)$ is an isomorphism. So we obtain the following theorem.

THEOREM 8.3. *Let $\tilde{H}^*$ be the functor that assigns to an L-foliation F the ring $H^*(P_F^1)$. Then the mapping $\psi = \psi(F) = (\pi_1^*)^{-1} \circ \Phi$: $H^*(L) \to H^*(P_F^1)$ is a homomorphism of the cohomology ring of L into the ring of characteristic classes in the category of L-foliations with values in $\tilde{H}^*$.*

Let H be a closed subgroup of $G_0(L)$. By Remark 3.2 we obtain a homomorphism Φ_H: $H^*(L, H) \to H^*(P_F/H)$. If H is homotopy equivalent to $G_0(L)$, then the projection π_H: $P_F/H \to M$ is a homotopy equivalence, and π_H^*: $H^*(M) \to H^*(P_F/M)$ is an isomorphism. So we obtain the following theorem:

THEOREM 8.4. *Let H^* be the functor that assigns to an L-foliation F the ring $H^*(M)$, and let H be a closed subgroup of $G_0(L)$ that is homotopy equivalent to $G_0(L)$. Then $\psi_H = (\pi_H^*)^{-1} \circ \Phi_H$ is a homomorphism of $H^*(L, H)$ into the ring of characteristic classes in the category of L-foliations with values in H^*.*

REMARK 8.1. For H we can take, for example, the maximal compact subgroup $K(L)$ of $G_0(L)$.

§9. Characteristic classes in the category of Γ-foliations

In this section we construct characteristic classes of the so-called Γ-foliations

DEFINITION 9.1 (see [2]). Let Γ be a transitive pseudogroup on $\mathbf{R}^n$.

A *Γ-foliation $\mathcal{F}$ of codimension n on a finite-dimensional manifold M* is a covering $\{U_i\}$ together with submersions f_i: $U_i \to \mathbf{R}^n$ that satisfy the condition: for every point $x \in U_i \cap U_j$ there exists an element $g_{ij} \in \Gamma$ such that $f_i = g_{ij} \circ f_j$ in some open neighborhood of x.

The submersions f_i are called *local projections* of the foliation $\mathcal{F}$. The leaf $\mathcal{F}_x$ of $\mathcal{F}$ passing through x is defined in a neighborhood U_i by the equation $f_i\,(\mathcal{F}_x \cap U_i) = f_i(x)$.

We make the class of Γ-foliations into a category.

DEFINITION 9.2. Let F_1 and F_2 be Γ-foliations on the manifolds M and N, respectively. A *morphism* $F_1 \to F_2$ is a smooth mapping f: $M \to N$ that satisfies the following condition.

Let $x \in M$, and let g_1: $U \to \mathbf{R}^n$ and g_2: $V \to \mathbf{R}^n$ ($U \subset M$, $V \subset N$) be mappings of neighborhoods of x and $f(x)$ that belong to the atlas of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then there exists a diffeomorphism $\gamma \in \Gamma$ such that $\gamma \circ g_1 = g_2 \circ f$ in some neighborhood of x.

REMARK 9.1. *Let $\mathcal{F}$ be a* Γ-foliation on N. We call a mapping f: *$M \to N$ transversal to $\mathcal{F}$* if the mappings $f_i \circ f$ are submersions, where the f_i are the local projections of $\mathcal{F}$. Then we can define uniquely on M the structure of a Γ-foliation $f^*(F)$ in such a way that f is a morphism $f^*(F) \to F$.

DEFINITION 9.3. Foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on a manifold M are *integrally-homotopic* if there exists a foliation $\mathcal{F}$ on $M \times [0, 1]$ such that $\mathcal{F}_k = i_k^*(\mathcal{F})$, where $k = 0, 1$, and i_k: $M \to M \times [0, 1]$ is an embedding that takes x to $x \times k$.

We describe now in a general way the connection between our concepts of an L-foliation and a Γ-foliation.

Suppose that a covering $\{U_i\}$ and submersions f_i: $U_i \to \mathbf{R}^n$ define on an $(n + m)$-dimensional manifold M a Γ-foliation $\mathcal{F}$ of codimension n. There is a transitive pseudogroup $\Gamma_{\mathcal{F}}$ on M connected with $\mathcal{F}$. By definition, $\Gamma_{\mathcal{F}}$ consists of the local diffeomorphisms ψ: $U_\psi \to V_\psi$ for which $f_j \circ \psi = \gamma \circ f_i$, in some neighborhood of an arbitrary point $x \in U_\psi$, where $x \in U_i$, $\psi(x) \in U_j$, and $\gamma \in \Gamma$. It can be shown that $L_{abs}(\Gamma_{\mathcal{F}})$ is isomorphic to $W_m(L) \subset W_{n+m}$, where L is isomorphic to $L_{abs}(\Gamma)$. The arguments in §7 enable us to construct a $W_m(L)$-structure for $\Gamma_{\mathcal{F}}$, that is, an L-foliation. Thus, for every Γ-foliation $\mathcal{F}$ we have constructed an L-foliation F. On the other hand, if F is an L-foliation in the sense of Definition 6.8, then there exists an ordinary foliation $\mathcal{F}$ on M, defined in the following way. A vector field $\zeta \in \mathfrak{A}(M)$ is tangent to a leaf of $\mathcal{F}$ if and only if $\omega(\sigma(\zeta)) \in V_{n,m}$. It can be shown that if F satisfies Theorem 7.1, then $\mathcal{F}$ is a Γ-foliation, with $L_{abs}(\Gamma)$ isomorphic to L.

The morphisms in the categories of L-foliations and Γ-foliations also are compatible.

Constructions later in this section generalize a construction proposed by the authors in [3].

Let $\mathcal{F}$ be a Γ-foliation of codimension n on a manifold M. We first construct an infinite-dimensional manifold $P_{\mathcal{F}}$, which plays the same role for $\mathcal{F}$ as P_F did for F.

To do this we consider the infinite-dimensional manifold $S_{\mathcal{F}}$ of formal submersions s: $M \to \mathbf{R}^n$ that are constant on the leaves of $\mathcal{F}$. An exact definition of $S_{\mathcal{F}}$ is as follows.

Let u be a local submersion of a neighborhood of $x \in M$ into $\mathbf{R}^n$ for which: a) u is constant on the leaves of $\mathcal{F}$; b) $u(x) = 0$. The space of k-jets at x of such submersions is denoted by $J_x^k(\mathcal{F})$. We set $S_{\mathcal{F}}^k = \{x, j \mid x \in M, j \in J_x^k(\mathcal{F})\}$ and $S_{\mathcal{F}} = \text{proj lim} = S_{\mathcal{F}}^k$. The $S_{\mathcal{F}}$ can be regarded as the space of the fibration $\pi_{\mathcal{F}}$: $S_{\mathcal{F}} \to M$ where the projection $\pi_{\mathcal{F}}$ is induced by the projections $\{x, j\} \mapsto x$. We define a subfibration $\pi_{\mathcal{F}}$: $P_{\mathcal{F}} \to M$ of $\pi_{\mathcal{F}}$: $S_{\mathcal{F}} \to M$ as proj lim $P_{\mathcal{F}}^k$, where $P_{\mathcal{F}}^k \subset S_{\mathcal{F}}^k$ consists of the points $\{x, j\}$, for which j is a k-jet of a submersion of the form $\gamma \circ f_i$, where $\gamma \in \Gamma$. ($P_{\mathcal{F}}^k$ does not depend on the choice for x of the index i, since in some neighborhood of x for any i, l, we have $f_i = g_{il} f$, where $g_{il} \in \Gamma$.) Now π_1: $P_{\mathcal{F}} \to P_{\mathcal{F}}^1$ is a fibration with an affine fiber, and so π_1 establishes a homotopy equivalence between the infinite-dimensional manifold $P_{\mathcal{F}}$ and the finite-dimensional manifold $P_{\mathcal{F}}^1$.

REMARK 9.4. We now give alternative descriptions of $S_{\mathcal{F}}^1$ and $P_{\mathcal{F}}^1$. Let $T(\mathcal{F})$ denote the subfibration of $T(M)$ tangent to the leaf of $\mathcal{F}$. The $S_{\mathcal{F}}^1$ is diffeomorphic to the total space of the principal $GL(n, \mathbf{R})$-bundle $Q(\mathcal{F})$ associated with the vector-bundle $T(M)/T(\mathcal{F})$, and $P_{\mathcal{F}}^1$ is diffeomorphic to the total space of the principal $G_0^1(L(\Gamma))$-bundle associated with $T(M)/T(\mathcal{F})$.

Now we construct an $L(\Gamma)$-valued differential form $\omega_{\mathcal{F}}$ on $P_{\mathcal{F}}$ that satisfies the Maurer-Cartan formula $d\omega_{\mathcal{F}} = -1/2[\omega_{\mathcal{F}}, \omega_{\mathcal{F}}]$. (We recall that the Lie algebra $L(\Gamma)$ is formed by the formal Taylor series of Γ-fields at zero.) For this purpose we consider the manifold $P(\mathbf{R}^n) \subset S(\mathbf{R}^n)$ of formal jets at zero of local diffeomorphisms $\gamma \in \Gamma$. It is easy to see that the pair $(\mathbf{R}^n, P(\mathbf{R}^n))$ defines an $L(\Gamma)$-structure on $\mathbf{R}^n$. With a local submersion s: $U \to \mathbf{R}^n (U \subset U_i \subset M)$, which is of the form $\gamma \circ f_i$, we associate the diffeomorphism $\gamma (\gamma \in \Gamma)$. The correspondence $s \mapsto \gamma$ induces a mapping of formal jets and consequently a mapping $\tilde{f}_i$: $P_{\mathcal{F}}|_{\pi_{\mathcal{F}}^{-1}(U_i)} \to P(\mathbf{R}^n)$. On the principal $L(\Gamma)$-space $P(\mathbf{R}^n)$ we take the canonical $L(\Gamma)$-valued form ω that satisfies the Maurer-Cartan formula. Then there is an $L(\Gamma)$-valued form $(\tilde{f}_i)^*\omega$ on $P_{\mathcal{F}}|_{\pi_{\mathcal{F}}^{-1}(U_i)}$. Clearly, $(\tilde{f}_i)^*\omega$ satisfies the Maurer-Cartan formula.

LEMMA 9.1. The local forms $(\tilde{f}_i)^*\omega$ define a global form $\omega_{\mathcal{F}}$ on $P_{\mathcal{F}}$.

PROOF. We must show that the forms $(\tilde{f}_i)^*\omega$ and $(\tilde{f}_j)^*\omega$ coincide on the intersection $P_{\mathcal{F}}|_{\pi_{\mathcal{F}}^{-1}(U_i)} \cap P_{\mathcal{F}}|_{\pi_{\mathcal{F}}^{-1}(U_j)} = P_{\mathcal{F}}|_{\pi_{\mathcal{F}}^{-1}}(U_i \cap U_j)$.

Let $x \in U_i \cap U_j$, and let V be a neighborhood of x on which $f_i = g_{ij} f_j$, where g_{ij} belongs to Γ. Then $(\tilde{f}_j)^*\omega'|_{\pi_{\mathscr{F}}^{-1}(V)} = (\tilde{f}_i)^*\omega$, where $\omega' = (g_{ij}^\infty)^*\omega$. It follows from Lemma 4.1 that the diffeomorphisms g_{ij}^∞ preserve ω, and so $\omega = \omega'$.

It is easy to check that the manifold $P_{\mathscr{F}}$ and the form $\omega_{\mathscr{F}}$ coincide with the manifold P_F and the form ω_F of the preceding section if the $L(\Gamma)$-foliation F is constructed from the Γ-foliation $\mathscr{F}$ by the method indicated in §7. Therefore Theorems 8.1 – 8.4 remain valid if F and $\mathscr{F}$ are interchanged.

Thus, for every Γ-foliation $\mathscr{F}$ on M there is a characteristic homomorphism

$$\Psi: H^*(L(\Gamma)) \to H^*(P_{\mathscr{F}}^1),$$

and also a characteristic homomorphism $\psi_H: H^*(L(\Gamma), H) \to H^*(M)$, where H is any closed subgroup of $G_0(\Gamma)$ that is homotopy equivalent to $G_0(\Gamma)$.

REMARK 9.3. It is clear from the construction of $\omega_{\mathscr{F}}$ that $\omega_{\mathscr{F}}$ is a characteristic class in the category of Γ-foliations with values in the functor that assigns to the Γ-foliation $\mathscr{F}$ the complex of $L(\Gamma)$-valued differential forms on $P_{\mathscr{F}}$. Consequently, Theorems 8.1 –8.4 can be proved for Γ-foliations without using the results of §7.

REMARK 9.4. Clearly, the images of the characteristic homomorphisms $\Psi(\mathscr{F})$ and $\Psi_H(\mathscr{F})$ depend only on the integrable-homotopy type of $\mathscr{F}$.

§10. Some applications to foliations of a special kind

In this section we study the characteristic classes of L-foliations when L is one of the algebras listed in §5. At the same time we mention some known results about the cohomology of L.

1. $L = W_n$. There are two possible variants of a W_n-foliation.

a) *An arbitrary foliation F.* In this case $G_0(W_n) = G_0$, and the maximal compact subgroup $K(W_n)$ of $G_0(W_n)$ is isomorphic to O_n, where O_n is the orthogonal group of order n. (The group $GL(n, \mathbf{R}) = GL_n$ is embedded in G_0, since every linear transformation can be regarded as a formal diffeomorphism of $\mathbf{R}^n$, that is, as an element of G_0.)

For an arbitrary W_n-foliation F we have the characteristic homomorphism $\psi: H^*(W_n) \to H^*(Q(F))$. (We recall that $Q(F)$ is the principal GL_n-bundle associated with the vector bundle normal to the leaves of F.)

We also recall some known results of Gel'fand and Fuks on the cohomology of W_n (a detailed proof of the theorem of Gel'fand and Fuks can be found in the article by Godbillon [12]).

Let T_n denote the product of the exterior algebra $E[u_1, \ldots, u_n]$ in the generators $u_1, \ldots, u_n$ ($\dim u_i = 2i - 1$) and the algebra $P_n[c_1, \ldots, c_n]$, which is the factor algebra of the algebra $P[c_1, \ldots, c_n]$ of polynomials in the variables $c_1, \ldots, c_n$ ($\dim c_i = 2i$) by the ideal generated by the monomials of degree $> 2n$. We make T_n into a complex by setting $du_i = c_i$ $(i = 1, \ldots, n)$.

THEOREM 10.1 (Gel'fand and Fuks). There exists a mapping t from T_n into the complex $C(W_n)$ that induces a cohomology isomorphism.

A basis of $H^*(T_n)$ consists of the cohomology classes of the elements $u_{i_1} \wedge \ldots \wedge u_{i_r} \otimes c_{j_1} \ldots c_{j_s}$, where the indices $i_1, \ldots, i_r, j_1, \ldots, j_s$ satisfy the conditions $i_1 < \ldots < i_r$, $j_1 \leqslant \ldots \leqslant j_s$, $j_1 + \ldots + j_s \leqslant n$, $i_1 + j_1 + \ldots + j_s > n$, $i_1 \leqslant j_1$. For convenience we use the same letters to denote the elements of T_n and the images of these elements under t.

We do not describe the homomorphism t completely, but we only indicate the images of the c_i. To the formal vector field ζ we assign its linear part, $\theta(\zeta)$, which can be identified with an element of the algebra $\mathfrak{gl}(n, \mathbf{R})$. The so obtained cochain θ with values in $\mathfrak{gl}(n, \mathbf{R}) = \mathfrak{gl}_n$ determines a $\mathfrak{gl}_n$-connection on W_n (for a definition, see [8], [12]). We define the curvature Ω from the structural formula

$$\Omega = 1/2[\theta, \theta] + d\theta.$$

We consider the algebra $J(\mathfrak{gl}_n)$ of polynomials on $\mathfrak{gl}_n$ invariant under the adjoint representation of GL_n. Now $J(\mathfrak{gl}_n)$ is isomorphic to $P[\tilde{c}_1, \ldots, \tilde{c}_n]$, where $\tilde{c}_i$ is the term of degree $n - i$ of the variable λ in $\det\left(\lambda E + \frac{1}{2\pi i}\Omega\right)$, where E is the identity matrix and $\Omega \in \mathfrak{gl}_n$. The image of $c_i \in T_n$ under t is defined by the formula:

$$(10.1) \qquad c_i(\zeta_1, \ldots, \zeta_{2i}) = \\ = \frac{1}{(2i)!} \sum_{\tau} \operatorname{sign} \tau \cdot \tilde{c}_i (\Omega(\zeta_{\tau(1)}, \zeta_{\tau(2)}), \ldots, \Omega(\zeta_{\tau(2i-1)}, \zeta_{\tau(2i)})) \\ (\zeta_1, \ldots, \zeta_{2i} \in W_n).$$

Knowing the cohomology of W_n and applying the Serre-Hochschild spectral sequence, we can easily compute the cohomology of the complexes $C(W_n, GL_n)$ and $C(W_n, O_n)$.

THEOREM 10.2. *The homomorphism t, restricted to the subcomplex $P_n[c_1, \ldots, c_n]$, induces an isomorphism*

$$t^*: P_n[c_1, \ldots, c_n] \to H^*(W_n, GL_n).$$

THEOREM 10.3. *Let TO_n denote the tensor product $E[u_1, u_3, u_5, \ldots] \otimes P_n[c_1, \ldots, c_n]$ (the generators u have odd suffixes). Then t, restricted to the subcomplex TO_n, induces an isomorphism*

$$t^*: H^*(TO_n) \to H^*(W_n, O_n).$$

THEOREM 10.4. *Let $\mathfrak{O}_n$ denote the Lie algebra of the group O_n. Then*

$$H^*(W_n, \mathfrak{O}_n) = H^*(W_n, SO_n) = \begin{cases} H^*(TO_n) & \text{if } n \text{ is odd,} \\ H^*(TO_n)\ [\chi]/\chi^2 - c_n & \text{if } n \text{ is even.} \end{cases}$$

Here χ is the formal Euler class, which is defined by the formal $\mathfrak{O}_n$-connection on W_n by the same formulae as the usual Euler class. (We can define the $\mathfrak{O}_n$-connection on W_n by setting $\theta_{\mathfrak{O}}(\zeta) = 1/2(\theta(\zeta)-\theta^*(\zeta))$, where $\zeta \in W_n$.)

Let us compute where the cohomology class c_i is sent under the homomorphism $\Psi_{GLn}(F)$: $H^*(W_n, GL_n) \to H^*(M)$.

THEOREM 10.5. *Let $p_i(F)$ denote the i-th Pontryagin class of the bundle $P_F^1 = Q(F)$. Then $\Psi_{GLn}(F)(c_{2i}) = p_i(F)$.*

PROOF. The form $\theta_F = \Phi(\theta)$ defined on P_F is a base for the bundle π_2: $P_F \to P_F^2$. Thus, there exists a canonical form on P_F^2 with values in $\mathfrak{gl}_n$ which we also denote by θ_F. Let $\tilde{\theta}$ be an arbitrary connection on the bundle $Q(F) \to M$. There exists a unique section ρ: $P_F^1 \to P_F^2$ for which $\tilde{\theta} = \rho^*(\theta_F)$. It remains to observe that the cocycles c_{2i} and the Pontryagin cocycles $p_i(\tilde{\theta})$ constructed from $\tilde{\theta}$ are expressed in the same way in terms of the forms θ and $\tilde{\theta}$. Consequently, $p_i(\tilde{\theta}) = \rho^*(\Phi F)(c_{2i})$. Reducing the forms $p_i(\theta)$ and $\rho^*\Phi(F)(c_{2i})$ to M and passing to cohomology, we obtain $p_i(F) = \Psi_{GLn}(F)(c_{2i})$.

COROLLARY 10.1 (Bott's theorem). *For an arbitrary foliation F of codimension n the product of the Pontryagin classes of the bundle $Q(F)$ of degree $> 2n$ vanishes.*

For this follows from the fact that the product of classes $c_i \in H^{2i}(W_n, GL_n)$ of degree $> 2n$ vanishes.

Apart from the classes c_{2i}, which become Pontryagin classes of $Q(F)$, we have characteristic classes of the form $u_{i_1} \wedge \dots \wedge u_{i_r} \otimes c_{j_1} \dots c_{j_s}$, where not all the indices are zero. Examples are known of foliations for which these characteristic classes are non-zero. The first example is due to Godbillon and Vey [11]. In their paper they describe a foliation F of codimension 1 for which the characteristic class $\Omega(F) = \Psi(F)\ (u_1 \wedge c_1)$ is different from zero. Other examples are in [10].

Thurston [29] has proved that there are foliations F of codimension 1 on the 3-dimensional sphere for which the value of the class $\Omega(F)$ on a fundamental cycle takes any positive value. Hence, in particular $\Omega(F)$ is not preserved under deformations of F.

On the other hand, Heitsch [33] has noted that a characteristic class of the form $\Psi(w)$, where the cohomology class $w \in H^*(W_n, O_n)$ lies in the image of the homomorphism h_n^*: $H^*(W_{n+1}, O_{n+1}) \to H^*(W_n, O_n)$, induces a natural embedding h_n: $W_n \to W_{n+1}$ that is preserved under deformations of the foliation. For if F_s for $s \in [0, 1]$ is a smooth family of foliations of codimension n on M, then there is a foliation F of codimension $n + 1$ on $M \times [0, 1]$. It is not difficult to check that the diagram

$$\begin{array}{ccc} H^*(W_{n+1}, O_{n+1}) & \xrightarrow{h_n^*} & H^*(W_n, O_n) \\ \downarrow{\scriptstyle \Psi(F)} & & \downarrow{\scriptstyle \Psi(F_s)} \\ H^*(M \times [0, 1]) & \xrightarrow{i_s^*} & H^*(M) \end{array}$$

commutes for every s, where i_s: $M \to M \times [0, 1]$ is an embedding that takes $x \in M$ to $x \times s \in M \times [0, 1]$. The required statement follows from this.

We now give an example of a foliation F_n of codimension n whose "generalized" Godbillon–Vey class $\Omega(F_n) = \Psi(F_n)\ (u_1 \wedge \cdots \wedge u_n \otimes c_1^n)$ is different from zero. (In the case $n = 1$, our example is the same as that in [11].)

EXAMPLE 10.1. Let SL_{n+1} be the group of unimodular matrices of order $(n + 1)$, $\mathfrak{sl}_{n+1}$ its Lie algebra, and Γ a discrete subgroup of SL_{n+1} such that the manifold $M = SL_{n+1}/\Gamma$ is compact. The subalgebra of $\mathfrak{sl}_{n+1}$ consisting of the matrices whose last row is zero defines on SL_{n+1} a distribution F_n of codimension n. This integrable distribution is invariant under the action of Γ (and also of SL_{n+1}), and consequently defines a foliation F_n on M. It can be verified that $\Omega(F_n)$ is different from zero.

b) An oriented foliation F. In this case $G_0(W_n) = G_0^+$, where G_0^+ is the group of formal jets of diffeomorphisms that preserve a fixed orientation of $\mathbf{R}^n$, and $K(W_n) = SO_n$, the special orthogonal group. The only difference from case a) is that for even n we obtain a new characteristic class $\Psi_{SO_n}(\chi)$, the Euler class of the oriented bundle $Q(F)$.

2. $L = W_n^{\mathbf{C}}$. A $W_n^{\mathbf{C}}$-foliation F is called a *complex analytic foliation of complex codimension n*. We first find the cohomology of $W_n^{\mathbf{C}}$. The computations of Gel'fand and Fuks carry over without change to $C(W_n^{\mathbf{C}}) \otimes \mathbf{C}$. Let $T_n^{\mathbf{C}}$ denote the tensor product

$$E^{\mathbf{C}}[u_1, \ldots, u_n] \otimes_{\mathbf{C}} P_n^{\mathbf{C}}[c_1, \ldots, c_n] \otimes_{\mathbf{C}} E^{\mathbf{C}}[\bar{u}_1, \ldots, \bar{u}_n] \otimes_{\mathbf{C}} P_n^{\mathbf{C}}[\bar{c}_1, \ldots, \bar{c}_n],$$

where $E^{\mathbf{C}}$ is the exterior algebra over $\mathbf{C}$, and $P_n^{\mathbf{C}}$ the factor algebra of the ring of polynomials with complex coefficients in n variables by the ideal generated by the monomials of degree $> 2n$. We make $T_n^{\mathbf{C}}$ into a complex by setting $du_i = c_i$ and $d\bar{u}_i = \bar{c}_i$.

THEOREM 10.6. *There exists a homomorphism $t^{\mathbf{C}}$ from $T_n^{\mathbf{C}}$ into $C(W_n^{\mathbf{C}}) \otimes \mathbf{C}$ that induces a homology isomorphism.*

The images of the c_k under this homomorphism are the cocycles defined by (10.1), with Ω replaced by the formal curvature constructed from the natural $\mathfrak{gl}(n, \mathbf{C})$-connection in $W_n^{\mathbf{C}}$ and the polynomials $\tilde{c}_k$ on the coefficients of degree $(n - k)$ with respect to λ in $\det\left(\lambda E + \frac{1}{2\pi i}\Omega\right)$. The cocycles $t^{\mathbf{C}}(\bar{c}_i)$ and $t^{\mathbf{C}}(\bar{u}_i)$ are conjugate to $t^{\mathbf{C}}(c_i)$ and $t^{\mathbf{C}}(u_i)$.

It can be shown that the cochains $u_i - \bar{u}_i$ belong to the complex $C(W_n^{\mathbf{C}}, U_n)$, where U_n is the unitary group of order n.

THEOREM 10.7. *Let $T^{\mathbf{C}}(U_n)$ denote the tensor product $P_n^{\mathbf{C}}[c_1, \ldots, c_n] \otimes_{\mathbf{C}} E^{\mathbf{C}}(u_1 - \bar{u}_1, \ldots, u_n - \bar{u}_n) \otimes_{\mathbf{C}} P_n^{\mathbf{C}}[\bar{c}_1, \ldots, \bar{c}_n]$. The homomorphism $t^{\mathbf{C}}$, restricted to the subcomplex $T^{\mathbf{C}}(U_n)$, induces an isomorphism $(t^{\mathbf{C}})^*: H^*(T^{\mathbf{C}}(U_n)) \to H^*(W_n, U_n) \otimes \mathbf{C}$.*

Now we return to the characteristic classes of complex analytic foliations.

For the $W_n^{\mathbf{C}}$-foliation F on the manifold M we have the characteristic homomorphism $\Psi(F)$: $H^*(W_n^{\mathbf{C}}) \to H^*(Q(F))$. (Here $Q(F)$ is the principal $GL(n, \mathbf{C})$-bundle associated with the complex vector bundle normal to the leaves of the complex foliation F.) Moreover, we have a homomorphism $\Psi_{U_n}(F)$: $H^*(W_n^{\mathbf{C}}, U_n) \to H^*(M)$. (The unitary group U_n is the maximal compact subgroup of the group $G_0(W_n^{\mathbf{C}})$ of formal analytic isomorphisms of $\mathbf{C}^n$.)

THEOREM 10.8. *Let $c_i(F)$ denote the i-th Chern class of the complex bundle $Q(F)$. Then*

$$\Psi_{U_n}(F)(c_i) = c_i(F).$$

The proof is analogous to that of Theorem 10.5.

COROLLARY 10.2 (Complex variant of Bott's theorem). For a complex analytic foliation F of complex codimension n the Chern classes of $Q(F)$ of degree $> 2n$ vanish.

EXAMPLE 10.2. Let M be an n-dimensional complex analytic manifold. There exists a natural $W_n^{\mathbf{C}}$-structure on M (see Example 4.2). Therefore we have a homomorphism Ψ: $H^*(W_n^{\mathbf{C}}) \to H^*(Q)$. (Here Q is the principal $GL(n, \mathbf{C})$-bundle associated with the complex tangent bundle $T(M)$.) Let us describe the homomorphism Ψ explicitly.

With this aim we consider the classifying map f: $M \to BU_n$ of M into the classifying space BU_n. By the theorem on cellular approximation we may assume that $f(M) \subset [BU_n]_{2n}$, where $[BU_n]_{2n}$ denotes the $2n$-skeleton of the cell complex BU_n. Let p: $E_{2n} \to [BU_n]_{2n}$ be the restriction of the universal principal $GL(n, \mathbf{C})$-bundle p: $E \to BU_n$ to $[BU_n]_{2n}$. The f induces a mapping $\tilde{f}$: $Q \to E_{2n}$. Let $H^*_{\mathbf{C}}(W_n^{\mathbf{C}})$ be the subring of $H^*(W_n^{\mathbf{C}}) \otimes \mathbf{C}$, that is the image under $t^{\mathbf{C}}$ of the cohomology ring of the subcomplex $E^{\mathbf{C}}[u, \dots, u_n] \otimes_{\mathbf{C}} P_n^{\mathbf{C}}[c_1, \dots, c_n]$ of $T_n^{\mathbf{C}}$.

THEOREM 10.9. *The image of $H^*_{\mathbf{C}}(W_n^{\mathbf{C}})$ under Ψ coincides with the image of the cohomology ring of E_{2n} under $\tilde{f}$, that is,* $\Psi(H^*_{\mathbf{C}}(W_n^{\mathbf{C}})) = \tilde{f}^*(H^*(E_{2n}, \mathbf{C}))$.

The proof of this theorem is left as an exercise.

REMARK 10.1. The natural projection $S^{\mathbf{C}}(M) \to Q$ that sends a formal analytic coordinate system on M to its 1-jet is a homotopy equivalence. Thus, in terms of the isomorphism $H^*(S^{\mathbf{C}}(M)) \approx H^*(Q)$, Theorem 10.9 describes in explicit form the homomorphism Φ: $H^*(W_n^{\mathbf{C}}) \to H^*(S^{\mathbf{C}}(M))$.

3. $L = D_{2n}$. The D_{2n}-foliation F is called Hamiltonian.

For a Hamiltonian foliation F we have the characteristic homomorphisms $\Psi(F)$: $H^*(D_{2n}) \to H^*(P_F^1)$ and $\Psi_{SP_{2n}}(F)$: $H^*(D_{2n}, SP_{2n}) \to H^*(M)$, where SP_{2n} is the symplectic group, regarded as a subgroup of $G_0(D_{2n})$. The study of the characteristic classes of Hamiltonian foliations connected with the homomorphism Ψ is beset by difficulties, because the cohomology of D_{2n} has not been computed completely. (A partial computation of this cohomology can be found in [9], for example.) However, even what little is known allows us to obtain new results on Hamiltonian foliations.

Here is the information we have on the cohomology of D_{2n}.

a) The cohomology of D_{2n} up to dimension $2n$ inclusive is generated by powers of a two-dimensional class $\gamma \in H^2(D_{2n})$. The class γ is defined by a cocycle $\gamma(\zeta, \eta) = \alpha(\zeta, \eta)(0)$, where $\alpha = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$, and $\zeta, \eta \in D_{2n}$.

b) D_{2n} has the connections $\mathfrak{sp}_{2n}$ and $\mathfrak{u}_n$, which are constructed like the $\mathfrak{gl}_{2n}$- and $\mathfrak{O}_{2n}$-connections of W_n. From these connections we can determine the formal Pontryagin classes p_i and Chern classes c_i in $H^*(D_{2n}, U_n)$. (Here $\mathfrak{sp}_{2n}$ and $\mathfrak{u}_n$ are the Lie algebras of the Lie groups SP_{2n} and U_n, respectively.)

All of the relations in $H^*(D_{2n}, U_n)$ between the formal Chern and Pontryagin classes are consequences of the relations in the rings of invariant polynomials $J(\mathfrak{u}_n)$ and $J(\mathfrak{sp}_{2n})$, and also of the Bott relations: $p_1^{i_1} \ldots p_r^{i_r} = 0$ if and only if $2(i_1 + \ldots + ri_r) > n$.

c) The embedding $D_{2n} \to W_{2n}$ induces a homomorphism $C(W_{2n}, \mathfrak{O}_{2n}) \to$ $\to C(D_{2n}, \mathfrak{u}_n)$. This homomorphism takes the cochains $u_i \in C(W_{2n}, \mathfrak{O}_{2n})$ (i odd) to zero.

d) Besides the classes of the form $u_{i_1} \wedge \ldots \wedge u_{i_r} \otimes c_{j_1} \ldots c_{j_s}$, which depend only on the 2-jets of vector fields, there are classes in the cohomology of D_{2n} that depend on jets of order greater than 2. For example, the class $p \in H^7(D_2)$ in [9] depends on a jet of order 4.

We can conclude from a) and b) that for a regular D_{2n}-structure $(M, P(M))$, in particular, for a Hamiltonian manifold M, the homomorphism Ψ_{U_n}: $H^*(D_{2n}, U_n) \to H^*(M)$ gives only the classical characteristic classes: the Pontryagin and Chern classes of the tangent bundle $T(M)$, and also the canonical form $\alpha = \Psi(\gamma)$. Apart from Bott's theorem for Hamiltonian foliations, the following results are consequences of c).

THEOREM 10.10. *A foliation F is Hamiltonian (or even integrably homotopic to a Hamiltonian foliation) only if the characteristic classes of the form $\Psi(F)$ $(u_{i_1} \wedge \ldots \wedge u_{i_r} \otimes c_{j_1} \ldots c_{j_s})$ vanish if at least one index i_t is odd.*

COROLLARY 10.3. *The foliation F_{2n} in example 10.1 is not integrably homotopic to any Hamiltonian foliation. In particular, the distribution F_{2n} cannot be defined by a closed 2-form.*

We do not know of any examples of Hamiltonian foliations with nontrivial characteristic classes $\Psi(p)$.

Hamiltonian foliations admit of a simple description by changing from the language of L-foliations to that of Γ-foliations.

Let $\Gamma(D_{2n})$ be a pseudogroup generated by local vector fields that belong to the algebra D_{2n} (that is, $\Gamma(D_{2n})$ is a pseudogroup of local diffeomorphisms of $\mathbf{R}^{2n}$ that preserve the 2-form $\alpha = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$).

In the language of Γ-foliations, a *Hamiltonian* foliation is a $\Gamma(D_{2n})$-foliation.

THEOREM 10.11. *The category of Hamiltonian foliations of codimension $2n$ is equivalent to the category whose objects are pairs (M, ω), where M is a finite-dimensional manifold and ω is a closed 2-form of constant rank $2n$ on M, and whose morphisms $(M, \omega) \to (M', \omega')$ are smooth maps $f: M \to M'$ such that $\omega = f^*(\omega')$.*

PROOF. 1) Let $F = \{u_i, f_i\}$ be a Hamiltonian foliation on M. The local projections $f_i: u_i \to \mathbf{R}^{2n}$ induce on U_i a 2-form $\omega_i = f_i^*\alpha$. The forms ω_i agree on the intersections $U_i \cap U_j$ and define a global form ω on M.

2) Given a pair (M, ω), we define a distribution F on M by the formula: $\zeta \in F \Leftrightarrow \iota_\zeta\omega = 0$ $(\zeta \in T(M))$. It is easy to check that F is integrable and defines a Hamiltonian foliation.

Clearly, the mappings described above agree with the morphisms in both categories, and establish their equivalence.

4. $L = A_n$. An A_n-foliation F is said to be *without divergence.* In this case there are characteristic homomorphisms $\Psi(F)$: $H^*(A_n) \to H^*(P_F^1)$ and $\Psi_{SL_n}(F)$: $H^*(A_n, SL_n) \to H^*(M)$, where SL_n is the group of unimodular matrices, regarded as a subgroup of $G_0(A_n)$. Although the ring $H^*(A_n)$ (like $H^*(D_{2n})$ has not been computed, we can obtain results for foliations without divergence analogous to those for Hamiltonian foliations in the previous example. We give a simple description of the category of foliations without divergence.

THEOREM 10.12. *The category of foliations of codimension n without divergence is equivalent to the category of pairs (M, ω), where M is a manifold and ω is a closed n-form on M that is constant of rank n. (The morphisms $(M, \omega) \to (M', \omega')$ are defined as in Theorem* 10.11.).

5. $L = K_n$. A K_n-foliation F is called *contact.* In this case there is a homomorphism $\Psi(F)$: $H^*(K_n) \to H^*(P_F^1)$. The ring $H^*(K_n)$ has also not been computed completely. However, in contrast to the algebras A_n and D_{2n} we can prove that $H^*(K_n)$ is finite-dimensional. Moreover, a direct computation of the cohomology of the algebras K_1, K_2, K_3 $(K_1 = W_1, K_2 = W_{1,1})$ shows that, at least for small n, all the cohomology classes of the contact algebras K_n can be represented by cocycles that depend only on the 2-jets of formal vector fields.

Contact foliations admit of a pretty description. We recall that every 1-form ω on M defines a distribution F by the formula $\zeta \in F \Leftrightarrow$ $\Leftrightarrow \iota_\zeta\omega = 0$ $(\zeta \in T(M))$. We say that ω is a form of class $2k$ if $(d\omega)^k$ vanishes nowhere, but $\omega \wedge (d\omega)^k \equiv 0$. Similarly, we say that ω is a form of class $2k + 1$ if $\omega \wedge (d\omega)^k$ vanishes nowhere, but $(d\omega)^{k+1} = 0$.

THEOREM 10.13. *The category of contact foliations of codimension n is equivalent to the category of pairs (M, F), where F is a distribution of codimension 1 on the manifold M that is given locally by a 1-form of class n. (A morphism $(M, F) \to (M', F')$ is a smooth mapping $f: M \to M'$ such that $F = f^*F'$.)*

The proof is analogous to that of Theorem 10.11.

§11. Characteristic classes in the category of principal G-bundles

In this section we show how to construct within the framework of our theory characteristic classes in the category of vector bundles and the category of principal G-bundles.

Let p: $E \to M$ be an n-dimensional real vector bundle over an m-dimensional manifold M. We denote by $P(E)$ the submanifold of E consisting of the formal jets at $0 \in \mathbf{R}^m \times \mathbf{R}^n$ of local isomorphisms of the bundle p: $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ to the bundle p: $E \to M$. It is not difficult to check that the pair $(E, P(E))$ is a $GL_n(W_m)$-structure. (The definition of the algebra $G(W_n)$, where G is a finite-dimensional Lie group, can be found in Example 5 of §5.) The algebra $GL_n(W_m)$ has a $\mathfrak{gl}_n$-connection θ, which assigns to an element of the form $X + g$ the free term of the formal power series g.

By means of the formal curvature Ω constructed from θ (10.1) defines cocycles $c_i \in C^{2i}(GL_n(W_m), GL_n \times GL_m)$. We also denote by c_i the cohomology classes corresponding to the cocycles c_i.

THEOREM 11.1. *Let p_i denote the i-th Pontryagin class of the vector bundle E. Then*

$$\Psi_{GL_n \times GL_m}(E)(c_{2i}) = p_i.$$

The proof is analogous to that of Theorem 10.5.

The construction above carries over without change to the case of a principal G-bundle p: $E \to M$. In this case we obtain a $G(W_m)$-structure $(E, P(E))$. To avoid obtaining only characteristic classes identically equal to zero, we assume that G is compact. Then the ring $J(\mathfrak{g})$ of G-invariant polynomials on $\mathfrak{g}$ is isomorphic to $P[c_1, \ldots, c_N]$. (The degree of the generator c_i is $2k_i$.) With the help of the $\mathfrak{g}$-connection on $G(W_m)$ and of the polynomials $c_i (i = 1, \ldots, N)$, we construct by (10.1) cocycles $c_i \in C^{2k_i}(G(W_m), G \times GL_m)$. The c_i define classes $c_i \in H^{2k_i}(G(W_m), G \times GL_m)$.

THEOREM 11.2. *Let $c_i(E)$ denote the values of the generators of the ring of real characteristic classes of the category of bundles p: $E \to M$. Then*

$$\Psi_{G \times GL_m}(E)(c_i) = c_i(E).$$

The proof is analogous to that of Theorem 10.5.

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