Adjoint Logic

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Abstract

We show that multiple intuitionistic logics with varying structural properties among weakening and contraction can be combined conservatively by adjoint pairs of modal operators. The resulting *adjoint logic* lies at the confluence of the ideas behind Benton's LNL and Nigam and Miller's subexponentials. We provide three different formulations of adjoint logic and show their equivalence: one with explicit structural rules, a second with implicit structural rules, and a third with focused rules. The first two provide the foundation for proofs-as-programs interpretations, while the third is well-suited for logic programming and logical frameworks.

We show that we can directly embed a number of previously proposed intuitionistic logics of interest in computer science such as linear logic, affine logic, strict logic, normal judgmental S4, lax logic, LNL, and normal intuitionistic subexponential linear logic. An interesting property of these embeddings is that proof-theoretic properties of the individual logics such as cut elimination, identity expansion, and focusing are immediate consequences of the corresponding theorems in adjoint logic.

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1 Introduction

How do we combine logics? One approach is to encode a less expressive into a more expressive logic. This is the approach, for example, taken by Girard [10] who represents the usual intuitionistic implication $A \to B$ as linear implication $!A \multimap B$ through the use of the exponential modality $!A$ that controls weakening and contraction. The rules of the source logic then become derived or admissible rules in the target logic. If we are interested in the computational interpretation of proofs via proof reduction, we then have to reconsider or reconstruct their meaning via their translation. Similarly, if we are interested in logic programming we typically lose the direct computational interpretation of propositions that is based on the fine structure of proofs.

An alternative is to keep the original logics intact and provide modal operators we call *shifts* to switch between them. This is the approach taken in the seminal work by Benton [4] on LNL. In this approach the individual logics are directly *embedded* into each other, which preserves not just provability but also the fine structure of proofs and thereby their computational interpretations.

In this paper we restrict our attention to intuitionistic logics and, in particular, we take the verificationist perspective [9, 7] where the meanings of the logical connectives in each logic are defined by the left and right rules of the sequent calculus. Cut elimination and identity expansion are necessary to justify this point of view.

All logics we consider satisfy associativity and exchange among the antecedents, and may or may not satisfy weakening or contraction. We identify a logic by its *mode of truth* m and write $\sigma(m) \subseteq \{\mathsf{W}, \mathsf{C}\}$ for the structural rules satisfied by mode m. We use the same definition for the logical connectives at all modes. For example, $A_m \multimap_m B_m$ denotes implication, which could be linear ($\sigma(m) = \{\,\}$), structural ($\sigma(m) = \{\mathsf{W}, \mathsf{C}\}$), affine ($\sigma(m) = \{\mathsf{W}\}$), or strict ($\sigma(m) = \{\mathsf{C}\}$). We often drop the subscript on the logical connective when it can be uniquely determined from context. In addition, we allow a *preorder* $m \geq k$ between modes of truth. As in subexponential linear logic [19] it is subject to the requirement that $m \geq k$ implies that $\sigma(m) \supseteq \sigma(k)$. As in LNL [4], each pair of shifts $\uparrow_k^m A_k$ and $\downarrow_k^m A_m$ must form an adjunction, which is guaranteed by their left and right rules.

Adjoint logic (ADJ) is designed so it satisfies the following properties: (1) *Conservative extension*: a proposition is provable in each source logic if and only if it is provable in the combination. (2) *Preservation of proofs and proof reduction*: proofs in each source logic will remain valid in the combination. Then proof reductions (the source of operational interpretations) also remain sound. Due to the presence of cut, additional proofs may be available in the combination. (3) *Preservation of focusing*: Focused proofs [1], which provide the foundation of logic programming, logical frameworks [21], and theorem provers [6, 17], remain complete for the combined logic.

Adjoint logic allows us to directly embed, combine, and uniformly generalize a number of previously investigated logics, such as intuitionistic linear logic [10, 2], LNL [4], normal judgmental S4 [23], lax logic [8, 12, 23], normal intuitionistic subexponential linear logic [5, 13], and its fragments such as affine logic or strict logic. In each case we construct a particular partial order and decomposition of modal operators into shifts. The original logic can then be clearly identified as a syntactic fragment, with some new propositions that can be expressed directly.

Providing a unified presentation of multiple logics with varying structural properties has a long history. For example, Belnap [3] defined Display Logic with structural connectives that control inferences such as weakening or contraction. Significant coding is required to put logics into display form, which impacts both proof reduction and proof search and therefore operational interpretations of a logic. Among the most closely related work we find work on LU [11] and LKU [15]. LU uses three explicit polarities, carefully controlled structural rules, and multiple versions of right and left rules to achieve an integration of classical, intuitionistic, and linear logic. This does not have the flexibility or parsimony of our approach, but allows classical logic to be a easily recognizable fragment. The same is true for LKU, which is based on classical logic, four different polarities, and focusing from the start. ADJ satisfies focusing, but does not require it, which is important for its proofs-as-programs interpretation that we are currently investigating [24].

Finally, work by Licata et al. [16] elegantly generalizes Reed's first and unpublished definition of adjoint logic [25] by providing more structure to the adjunctions. Briefly, the preorder of modes is replaced by a 2-category, which allows the faithful representation of additional examples (for example, all of judgmental S4 instead of just its normal fragment), including some constructions in homotopy type theory. They also provide a detailed categorical semantics. However, the independence principle upon which ADJ is based is only an emergent property of some mode structures, which in some cases leads to more complicated adequacy proofs. Also, they do not investigate focusing or polarization.

We begin with a variation of Reed's first and unpublished definition of adjoint logic [25] by using explicit structural rules and just a single pair of left and right rules for each of the logical connectives and shifts. This formulation allows an elegant proof of cut elimination, closely modeled upon Gentzen's original proof [9], using the rule of *multicut*. Cut elimination immediately yields a conservative extension result for the combined logic over all of its modes of truth. Next, we provide a formulation where structural rules remain implicit: that is, they are incorporated directly into the various rules of the calculus. This avoids explicating weakening and contraction that are implicit in some of the source logics we model, and also provides a stepping stone towards focusing. Finally, we provide a polarized and focused presentation of adjoint logic. Focusing fits neatly into the framework of shifts upon which adjoint logic is based by using shifts within a mode to represent polarization.

2 Adjoint Logic with Explicit Structural Rules (ADJ^E)

Adjoint logic (in all of the forms that we present it in) can be thought of as a schema to define particular logics. As described in Section 1, the schema is parameterized by a preorder of modes of truth m, along with a monotone map σ from this preorder into $\mathcal{P}(\{\mathsf{W}, \mathsf{C}\})$ assigning to each mode its set of structural properties. As a concession to simplicity of the presentation, in this paper we always allow exchange, although nothing stands in the way of an even more general framework [13]. This preorder of modes embodies the *declaration of independence*:

A proof of A_k may only depend on hypotheses B_m for $m \geq k$.

The form of a sequent is therefore

$$\Psi \vdash A_k \quad \text{where } \Psi \geq k$$

where Ψ is a collection of *antecedents* of the form $B^i_{m_i}$ with each $m_i \geq k$. This critical presupposition, abbreviated as $\Psi \geq k$, generalizes the common dyadic [1] or two-zone presentation of logics with modal operators [2] where the judgment of validity may not depend on hypotheses about truth [23]. Example 2 illustrates that for cut elimination to hold, the structural properties of the permissible antecedent modes must include those of the succedent mode.

The propositions at each mode are constructed uniformly, remaining within the same mode, except for the *shift operators* that move between modes. They are $\uparrow^m_k A_k$ (pronounced *up*), which is a proposition at mode m and requires $m \geq k$; and $\downarrow^\ell_m A_\ell$ (*down*), which is also a proposition at mode m, and which requires $\ell \geq m$.

At this point we can already write out the syntax of propositions.

$$A_m, B_m \quad ::= \quad p_m \mid A_m \multimap_m B_m \mid A_m \otimes_m B_m \mid \mathbf{1}_m \mid \oplus_{j\in J} A^j_m \mid \&_{j\in J} A^j_m \mid \uparrow^m_k A_k \mid \downarrow^\ell_m A_\ell$$

Here p_m stands for atomic propositions at mode m. Anticipating the needs of an operational interpretation (see [24] for a sketch), we have generalized internal and external choice to n-ary constructors parameterized by a finite index set J. With $J = \emptyset$ we recover $\top = \&_{j\in\emptyset}(\,)$ and $\mathbf{0} = \oplus_{j\in\emptyset}(\,)$ in any mode. The right and left rules in the sequent calculus defining the

$$\frac{}{A_m \vdash A_m}\ \mathsf{id} \qquad \frac{\Psi_1 \geq m \geq k \quad \Psi_1 \vdash A_m \quad \Psi_2, A_m \vdash C_k}{\Psi_1, \Psi_2 \vdash C_k}\ \mathsf{cut}$$

$$\frac{\mathsf{W} \in \sigma(m) \quad \Psi \vdash C_k}{\Psi, A_m \vdash C_k}\ \mathsf{weaken} \qquad \frac{\mathsf{C} \in \sigma(m) \quad \Psi, A_m, A_m \vdash C_k}{\Psi, A_m \vdash C_k}\ \mathsf{contract}$$

$$\frac{i \in J \quad \Psi \vdash A_m^i}{\Psi \vdash \oplus_{j\in J} A_m^j}\ \oplus R^i \qquad \frac{\Psi, A_m^j \vdash C_k \quad \text{for all } j \in J}{\Psi, \oplus_{j\in J} A_m^j \vdash C_k}\ \oplus L$$

$$\frac{\Psi \vdash A_m^j \quad \text{for all } j \in J}{\Psi \vdash \&_{j\in J} A_m^j}\ \& R \qquad \frac{i \in J \quad \Psi, A_m^i \vdash C_k}{\Psi, \&_{j\in J} A_m^j \vdash C_k}\ \& L^i$$

$$\frac{\Psi_1 \vdash A_m \quad \Psi_2 \vdash B_m}{\Psi_1, \Psi_2 \vdash A_m \otimes B_m}\ \otimes R \qquad \frac{\Psi, A_m, B_m \vdash C_k}{\Psi, A_m \otimes B_m \vdash C_k}\ \otimes L$$

$$\frac{\Psi, A_m \vdash B_m}{\Psi \vdash A_m \multimap B_m}\ \multimap R \qquad \frac{\Psi_1 \geq m \quad \Psi_1 \vdash A_m \quad \Psi_2, B_m \vdash C_k}{\Psi_1, \Psi_2, A_m \multimap B_m \vdash C_k}\ \multimap L$$

$$\frac{\mathsf{W} \in \sigma(\Psi)}{\Psi \vdash \mathbf{1}_m}\ \mathbf{1}R \qquad \frac{\Psi \vdash C_k}{\Psi, \mathbf{1}_m \vdash C_k}\ \mathbf{1}L$$

$$\frac{\Psi \geq m \quad \Psi \vdash A_m}{\Psi \vdash {\downarrow}_k^m A_m}\ {\downarrow}R \qquad \frac{\Psi, A_m \vdash C_\ell}{\Psi, {\downarrow}_k^m A_m \vdash C_\ell}\ {\downarrow}L$$

$$\frac{\Psi \vdash A_k}{\Psi \vdash {\uparrow}_k^m A_k}\ {\uparrow}R \qquad \frac{k \geq \ell \quad \Psi, A_k \vdash C_\ell}{\Psi, {\uparrow}_k^m A_k \vdash C_\ell}\ {\uparrow}L$$

Figure 1 Adjoint Logic with Explicit Structural Rules (ADJ^E).
We presuppose that the conclusion of each rule satisfies the declaration of independence and ensure, with conditions on modes, that the premises will, too.

logical connectives are the same for each mode and are complemented by the permissible structural rules.

In order to be able to describe side conditions for rules, we generalize $\sigma(m)$ to $\sigma(\Psi)$ for contexts Ψ.

▶ **Definition 1.** We define $\sigma(\Psi)$ inductively as follows:

$$\begin{array}{rcl} \sigma(\cdot) & = & \{\mathsf{W}, \mathsf{C}\} \\ \sigma(A_m) & = & \sigma(m) \\ \sigma(\Psi_1, \Psi_2) & = & \sigma(\Psi_1) \cap \sigma(\Psi_2) \end{array}$$

Intuitively, $\sigma(\Psi)$ is the smallest set of structural properties shared by all propositions in Ψ, so if $\mathsf{W} \in \sigma(\Psi)$, then every proposition in Ψ is subject to weakening, and if $\mathsf{C} \in \sigma(\Psi)$, then every proposition in Ψ is contractible.

2.1 Judgmental and structural rules

The rules for this first version of adjoint logic (ADJ^E) can be found in Figure 1. We begin with the judgmental rules of identity and cut, which express the connection between antecedents and succedents. Identity says that if we assume A_m we are allowed to conclude A_m. Cut says the opposite: if we can conclude A_m we are allowed to assume A_m *as long as the declaration of independence is respected.*

As is common for the sequent calculus, we read the rules in the direction of bottom-up proof construction. For the cut rule, this means we should assume that the conclusion $\Psi_1, \Psi_2 \vdash C_k$ is well-formed and, in particular, that $\Psi_1 \geq k$ and $\Psi_2 \geq k$. Therefore, if we check that $m \geq k$, then we know that the second premise, $\Psi_2, A_m \vdash C_k$, will also be well-formed. For the first premise to be well-formed, we need to check outright that $\Psi_1 \geq m$.

The structural rules of weakening and contraction just need to verify that the mode of the principal formula permits the rule.

2.2 Additive and multiplicative connectives

The logical rules defining the additive and multiplicative connectives are simply the linear rules for all modes, since we have separated out the structural rules. Except in one case, $\multimap L$, the well-formedness of the conclusion implies the well-formedness of all premises.

As for $\multimap L$, we know from the well-formedness of the conclusion that $\Psi_1 \geq k$, $\Psi_2 \geq k$, and $m \geq k$. These facts by themselves already imply the well-formedness of the second premise, but we need to check that $\Psi_1 \geq m$ in order for the first premise to be well-formed.

2.3 Shifts

The shifts represent the most interesting aspects of the rules. Recall that in $\uparrow_k^m A_k$ and $\downarrow_k^m A_m$ we require that $m \geq k$. We first consider the two rules for $\uparrow$. We know from the conclusion of the right rule that $\Psi \geq m$ and from the requirement of the shift that $m \geq k$. Therefore, as $\geq$ is transitive, $\Psi \geq k$ and the premise is always well-formed. This also means that this rule is *invertible*, an observation integrated into the focusing rules for system ADJ^F presented in Section 4.

From the conclusion of the left rule, we know $\Psi \geq \ell$, $m \geq \ell$, and $m \geq k$. This does not imply that $k \geq \ell$, which we need for the premise to be well-formed and thus needs to be checked. Therefore, this rule is non-invertible.

The downshift rules are constructed analogously, taking only the declaration of independence and properties of the preorder $\leq$ as guidance. Note that in this case the left rule is always applicable (that is, invertible), while the right rule is non-invertible.

The fact that the shift operators form an adjunction in an appropriate category was first observed by Benton [4] for LNL where $F\,X = \downarrow_{\mathsf{L}}^{\mathsf{U}} X_{\mathsf{U}}$ is shown to be the left adjoint to $G\,A = \uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}}$ (see Section 3.2 for more detail on interpreting LNL in ADJ), going between cartesian closed and symmetric monoidal categories. A more general categorical interpretation is given by Licata et al. [16]. A direct proof of the adjunction property is given in [24, A.3], by considering equivalence classes of proofs up to cut reductions, commuting conversions, and identity expansion.

► **Example 2** (Counterexample for independence)**.** Consider an instance with two modes $\mathsf{L} < \mathsf{U}$ where $\sigma(\mathsf{L}) = \emptyset$ and $\sigma(\mathsf{U}) = \{\mathsf{W}, \mathsf{C}\}$ and consider the following **faulty(!)** "proof" showing that contraction for linear propositions is derivable:

$$
\dfrac{
  \dfrac{
    \dfrac{}{A_{\mathsf{L}} \vdash A_{\mathsf{L}}}\;\mathsf{id}
  }{A_{\mathsf{L}} \vdash^{??} \uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}}}\;{\uparrow}R
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        A_{\mathsf{L}}, A_{\mathsf{L}} \vdash C_{\mathsf{L}}
      }{A_{\mathsf{L}}, \uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}} \vdash C_{\mathsf{L}}}\;{\uparrow}L
    }{\uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}}, \uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}} \vdash C_{\mathsf{L}}}\;{\uparrow}L
  }{\uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}} \vdash C_{\mathsf{L}}}\;\mathsf{contract}
}{A_{\mathsf{L}} \vdash C_{\mathsf{L}}}\;\mathsf{cut}
$$

The fallacy lies with the sequent marked $\vdash^{??}$ because it violates our declaration of independence: the succedent $\uparrow_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}}$ of mode U depends on an antecedent of mode L, and $\mathsf{L} \not\geq \mathsf{U}$.

If we wanted to blame a particular inference, it would be either cut, viewed bottom-up, or $\uparrow R$, viewed top-down. In our case, the bottom-up construction of this proof would fail because the condition $A_{\mathsf{L}} \geq \mathsf{U}$ of the cut rule is violated.

It is an immediate corollary that cut elimination fails if the declaration of independence is not enforced. For example, using the above faulty reasoning, we could prove $A_{\mathsf{L}} \vdash A_{\mathsf{L}} \otimes A_{\mathsf{L}}$, which in general has no cut-free proof.

2.4 Cut elimination

Because we have an explicit rule of contraction, cut elimination does not follow by a simple structural induction. However, we can follow Gentzen [9] and allow multiple copies of the same proposition to be removed by the cut, which then allows a structural induction argument. The generalized form of cut called *multicut* (see, for example, Negri and von Plato [18]) can remove $n \geq 0$ copies of a proposition, provided that the structural properties of that proposition allow it.

We define $\mu(S)$ for $S \subseteq \{\mathsf{W}, \mathsf{C}\}$ to be the set of permissible multiplicities corresponding to the set S of structural properties. This has the following definition:

$$\mu(\{\}) = \{1\} \qquad \mu(\{\mathsf{W}\}) = \{0, 1\} \qquad \mu(\{\mathsf{C}\}) = \mathbb{N} \setminus \{0\} \qquad \mu(\{\mathsf{W}, \mathsf{C}\}) = \mathbb{N}$$

With this, we can write down a general multicut rule where we allow cutting out n copies of A_m if $n \in \mu(\sigma(m))$:

$$\frac{\Psi_1 \geq m \geq k \quad n \in \mu(\sigma(m)) \quad \Psi_1 \vdash A_m \quad \Psi_2, A_m^n \vdash C_k}{\Psi_1, \Psi_2 \vdash C_k}\ \mathsf{cut}(n)$$

Note that the standard cut rule is the instance of the multicut rule where $n = 1$, and so proving multicut elimination for adjoint logic also yields cut elimination for the standard cut rule.

In order to distinguish proofs in ADJ^E from proofs in the other two systems that we will present, we use $\vdash_E$ for proofs in ADJ^E. Analogously, we will use $\vdash_I$ for proofs in ADJ^I (Section 3) and $\vdash_F$ for proofs in ADJ^F (Section 4). For the purposes of our cut admissibility and cut elimination proofs, we also introduce the notation $\Vdash_E$ for proofs in ADJ^E that do not use the cut rule.

▶ **Theorem 3** (Admissibility of multicut). *If $\Psi_1 \geq m \geq k$, $n \in \mu(\sigma(m))$, $\Psi_1 \Vdash_E A_m$, and $\Psi_2, A_m^n \Vdash_E C_k$, then $\Psi_1, \Psi_2 \Vdash_E C_k$.*

Sketch of proof. This follows straightforwardly by induction on the (lexicographically ordered) triple $(A_m, \mathcal{D}, \mathcal{E})$, where $\mathcal{D}$ is the proof that $\Psi_1 \vdash_E A_m$ and $\mathcal{E}$ is the proof that $\Psi_2, A_m^n \vdash_E C_k$. ◀

▶ **Theorem 4** (Cut elimination for ADJ^E). *If $\Psi \vdash_E A_m$, then $\Psi \Vdash_E A_m$.*

Proof. This follows from admissibility of multicut by induction over the proof that $\Psi \vdash_E A_m$, using admissibility of multicut to eliminate each cut as it is encountered. ◀

2.5 Identity Expansion

Identity expansion for this system is very standard in both its statement and its proof.

▶ **Theorem 5** (Identity Expansion). *If $\Psi \vdash_E A_m$, then there exists a proof that $\Psi \vdash_E A_m$ using identity rules only at atomic propositions, which is cut-free if the original proof is.*

Proof. We begin by proving that for any formula A_m, there is a cut-free proof that $A_m \vdash_E A_m$ using identity rules only at atomic propositions. This follows easily from an induction on A_m. Now, we arrive at the theorem by induction over the structure of the given proof that $\Psi \vdash_E A_m$. ◀

3 Adjoint Logic with Implicit Structural Rules (ADJ^I)

As a first step along the way to focusing, we present a system which removes some of the nondeterminism arising in proof search from the explicit structural rules, analogously to the *dyadic* system Σ_2 presented by Andreoli [1]. A side benefit of this system is that its implicit treatment of structural rules makes it better suited for embedding logics which similarly leave structural properties implicit.

As we allow modes to have only one of weakening and contraction and as we also have shifts built into the logic, we cannot take exactly the same approach as Andreoli. Most of the shift rules are straightforward to translate, but $\downarrow R$ requires some thought because of its restriction on the modes allowed in the context. In ADJ^E, we can weaken away any A_m with $\mathsf{W} \in \sigma(m)$ before applying $\downarrow R$ in order to satisfy that restriction. In order to match that behavior, we split the context into two pieces, Ψ_1 and Ψ_2, and require that $\Psi_1 \geq m$, while $\mathsf{W} \in \sigma(\Psi_2)$. This rule then corresponds to the ADJ^E proof which weakens everything in Ψ_2 and then applies $\downarrow R$. Weakening is otherwise easily handled at the leaves of the proof in a similar manner.

Contraction without weakening leads to most of the complication in this system. For each multiplicative rule with two premises ($\otimes R$, $\multimap L$, and cut), we split the context into three parts sending Ψ_1 to the first premise only, Ψ_3 to the second premise only, and Ψ_2 to both (and, of course, we require that Ψ_2 be contractible). The nondeterminism in choice of Ψ_2 allows us to propagate contractible propositions to precisely those premises where they will be needed. Similarly, using the notation of Definition 6, we allow for (but do not force) the principal formula to be kept after applying a left rule. These changes, along with the removal of the weakening and contraction rules, give us ADJ^I, as shown in Figure 2.

▶ **Definition 6.**
1. $(A_m)^?$ may always denote the empty context $(\cdot)$.
2. If $\mathsf{C} \in \sigma(m)$, then $(A_m)^?$ may denote A_m.

3.1 Equivalence of ADJ^I and ADJ^E

We now show the equivalence (in terms of provability) of ADJ^E and ADJ^I. Soundness is almost immediate, as the changes necessary to turn ADJ^E into ADJ^I are fairly minor. Completeness follows quickly from Lemma 8. An interesting (if unsurprising) feature of the translations used in both soundness and completeness is that cut-free proofs are taken to cut-free proofs, and so cut elimination can be transported from one system to the other. Similarly, the translations take identities at A_m to identities at A_m, and so identity expansion can also be transported from one system to the other. We therefore get for free that ADJ^I has both cut elimination and identity expansion as a result of its soundness and completeness proofs. Since the proofs are entirely standard we omit them here for the sake of brevity.

▶ **Theorem 7** (Soundness of ADJ^I)**.** *If $\Psi \vdash_I A_m$, then $\Psi \vdash_E A_m$.*

▶ **Lemma 8** (Admissibility of weakening and contraction for ADJ^I)**.**
1. *If $\Psi \vdash_I C_k$ and $\mathsf{W} \in \sigma(m)$, then $\Psi, A_m \vdash_I C_k$.*

$$\frac{\mathsf{W} \in \sigma(\Psi)}{\Psi, A_m \vdash A_m}\ \mathsf{id} \qquad \frac{\Psi_1, \Psi_2 \geq m \geq k \quad \mathsf{C} \in \sigma(\Psi_2) \quad \Psi_1, \Psi_2 \vdash A_m \quad \Psi_2, \Psi_3, A_m \vdash C_k}{\Psi_1, \Psi_2, \Psi_3 \vdash C_k}\ \mathsf{cut}$$

$$\frac{i \in J \quad \Psi \vdash A^i_m}{\Psi \vdash \oplus_{j \in J} A^j_m}\ \oplus R^i \qquad \frac{\Psi, (\oplus_{j \in J} A^j_m)^?, A^j_m \vdash C_k \quad \text{for all } j \in J}{\Psi, \oplus_{j \in J} A^j_m \vdash C_k}\ \oplus L^{(*)}$$

$$\frac{\Psi \vdash A^j_m \quad \text{for all } j \in J}{\Psi \vdash \&_{j \in J} A^j_m}\ \& R \qquad \frac{i \in J \quad \Psi, (\&_{j \in J} A^j_m)^?, A^i_m \vdash C_k}{\Psi, \&_{j \in J} A^j_m \vdash C_k}\ \& L^i$$

$$\frac{\mathsf{C} \in \sigma(\Psi_2) \quad \Psi_1, \Psi_2 \vdash A_m \quad \Psi_2, \Psi_3 \vdash B_m}{\Psi_1, \Psi_2, \Psi_3 \vdash A_m \otimes B_m}\ \otimes R \qquad \frac{\Psi, (A_m \otimes B_m)^?, A_m, B_m \vdash C_k}{\Psi, A_m \otimes B_m \vdash C_k}\ \otimes L$$

$$\frac{\Psi, A_m \vdash B_m}{\Psi \vdash A_m \multimap B_m}\ \multimap R$$

$$\frac{\Psi_1, \Psi_2 \geq m \quad \mathsf{C} \in \sigma(\Psi_2) \quad \Psi_1, \Psi_2, (A_m \multimap B_m)^? \vdash A_m \quad \Psi_2, \Psi_3, (A_m \multimap B_m)^?, B_m \vdash C_k}{\Psi_1, \Psi_2, \Psi_3, A_m \multimap B_m \vdash C_k}\ \multimap L$$

$$\frac{\mathsf{W} \in \sigma(\Psi)}{\Psi \vdash \mathbf{1}_m}\ \mathbf{1}R \qquad \frac{\Psi, (\mathbf{1}_m)^? \vdash C_k}{\Psi, \mathbf{1}_m \vdash C_k}\ \mathbf{1}L$$

$$\frac{\Psi_1 \geq m \quad \mathsf{W} \in \sigma(\Psi_2) \quad \Psi_1 \vdash A_m}{\Psi_1, \Psi_2 \vdash {\downarrow}^m_k A_m}\ {\downarrow}R \qquad \frac{\Psi, ({\downarrow}^m_k A_m)^?, A_m \vdash C_\ell}{\Psi, {\downarrow}^m_k A_m \vdash C_\ell}\ {\downarrow}L$$

$$\frac{\Psi \vdash A_k}{\Psi \vdash {\uparrow}^m_k A_k}\ {\uparrow}R \qquad \frac{k \geq \ell \quad \Psi, ({\uparrow}^m_k A_k)^?, A_k \vdash C_\ell}{\Psi, {\uparrow}^m_k A_k \vdash C_\ell}\ {\uparrow}L$$

Figure 2 Adjoint Logic with Implicit Structural Properties (ADJ^I).
We presuppose that the conclusion of each rule satisfies the declaration of independence and ensure, with conditions on modes, that the premises will, too. $(A_m)^?$ denotes an optional antecedent if $C \in \sigma(m)$.
$(*)$ All premises of $\oplus L$ must uniformly retain or delete $(\,)^?$.

2. *If* $\Psi, A_m, A_m \vdash_I C_k$ *and* $\mathsf{C} \in \sigma(m)$*, then* $\Psi, A_m \vdash_I C_k$.

▶ **Theorem 9** (Completeness of ADJ^I)**.** *If* $\Psi \vdash_E A_m$ *then* $\Psi \vdash_I A_m$.

3.2 Logic Embeddings

We now illustrate how adjoint logic can be used to embed various logics.

▶ **Example 10** (Linear logic)**.** We obtain intuitionistic linear logic [10, 2] by using two modes, U (for *unrestricted* or structural) and L (for *linear*) with $\mathsf{U} > \mathsf{L}$. Moreover, $\sigma(\mathsf{U}) = \{\mathsf{W}, \mathsf{C}\}$ and $\sigma(\mathsf{L}) = \{\,\}$, and the structural layer contains only shifted propositions.

$$\begin{array}{lcl} A_{\mathsf{U}} & ::= & {\uparrow}^{\mathsf{U}}_{\mathsf{L}} A_{\mathsf{L}} \\ A_{\mathsf{L}}, B_{\mathsf{L}} & ::= & p_{\mathsf{L}} \mid A_{\mathsf{L}} \multimap B_{\mathsf{L}} \mid A_{\mathsf{L}} \otimes B_{\mathsf{L}} \mid \mathbf{1} \mid \oplus_{j \in J} A^j_{\mathsf{L}} \mid \&_{j \in J} A^j_{\mathsf{L}} \mid {\downarrow}^{\mathsf{U}}_{\mathsf{L}} A_{\mathsf{U}} \end{array}$$

In this representation the exponential modality is decomposed into shift modalities $!A_{\mathsf{L}} = {\downarrow}^{\mathsf{U}}_{\mathsf{L}} {\uparrow}^{\mathsf{U}}_{\mathsf{L}} A_{\mathsf{L}}$. We do not state an explicit correctness theorem because it follows from the embedding of LNL (Theorem 12) and Benton's results [4].

▶ **Example 11** (LNL). We obtain LNL [4] just like linear logic with two modes $\mathsf{U} > \mathsf{L}$, but we populate the unrestricted layer with additional propositions, where we write $\times = \otimes_{\mathsf{U}}$ and $\rightarrow = \multimap_{\mathsf{U}}$.

$$\begin{array}{lcl} A_{\mathsf{U}}, B_{\mathsf{U}} & ::= & p_{\mathsf{U}} \mid A_{\mathsf{U}} \rightarrow B_{\mathsf{U}} \mid A_{\mathsf{U}} \times B_{\mathsf{U}} \mid 1_{\mathsf{U}} \mid {\uparrow}_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{L}} \\ A_{\mathsf{L}}, B_{\mathsf{L}} & ::= & p_{\mathsf{L}} \mid A_{\mathsf{L}} \multimap B_{\mathsf{L}} \mid A_{\mathsf{L}} \otimes B_{\mathsf{L}} \mid \mathbf{1}_{\mathsf{L}} \mid {\downarrow}_{\mathsf{L}}^{\mathsf{U}} A_{\mathsf{U}} \end{array}$$

Benton's notation for shifts is $F = {\downarrow}_{\mathsf{L}}^{\mathsf{U}}$ and $G = {\uparrow}_{\mathsf{L}}^{\mathsf{U}}$. Our formulation then combines the various versions of the rules by combining the two contexts, using the declaration of independence instead to force that unrestricted succedents depend only on unrestricted antecedents. A small difference arises only in the $\times$*-left* rules where our version has both components in the premise, which is, of course, logically equivalent to LNL in the presence of weakening and contraction.

As LNL uses explicit structural rules, it is more direct to embed LNL into ADJ^E than into ADJ^F, and so our theorem is stated using ADJ^E. It is, of course, equivalent to use ADJ^I.

▶ **Theorem 12.** *If we let τ embed propositions of LNL into the instance of adjoint logic described above, then*
1. $\Theta \vdash_{\mathcal{C}} X$ *in LNL iff* $\tau(\Theta) \vdash_E \tau(X)$.
2. $\Theta; \Gamma \vdash_{\mathcal{L}} A$ *in LNL iff* $\tau(\Theta), \tau(\Gamma) \vdash_E \tau(A)$.

▶ **Example 13** (Judgmental S4). The judgmental modal logic S4 [23] arises from two modes V (validity) and U (truth) with $\mathsf{V} > \mathsf{U}$. The declaration of independence here expresses that *validity is categorical with respect to truth*—that is, a proof of A_{V} may not depend on any hypotheses of the form B_{U}. Previously, this had been enforced by segregating the antecedents into two zones and managing their dependence accordingly.

$$\begin{array}{lcl} A_{\mathsf{V}} & ::= & {\uparrow}_{\mathsf{U}}^{\mathsf{V}} A_{\mathsf{U}} \\ A_{\mathsf{U}}, B_{\mathsf{U}} & ::= & p_{\mathsf{U}} \mid A_{\mathsf{U}} \multimap B_{\mathsf{U}} \mid A_{\mathsf{U}} \otimes B_{\mathsf{U}} \mid \mathbf{1} \mid \oplus_{j \in J} A_{\mathsf{U}}^j \mid \&_{j \in J} A_{\mathsf{U}}^j \mid {\downarrow}_{\mathsf{U}}^{\mathsf{V}} A_{\mathsf{V}} \end{array}$$

Analogous to the encoding of linear logic, we only need to allow ${\uparrow}_{\mathsf{U}}^{\mathsf{V}} A_{\mathsf{U}}$ in the validity layer. Under that interpretation, we encode $\Box A_{\mathsf{U}} = {\downarrow}_{\mathsf{U}}^{\mathsf{V}} {\uparrow}_{\mathsf{U}}^{\mathsf{V}} A_{\mathsf{U}}$, which is entirely analogous to the representation of $!A$ in linear logic.

The adjoint reconstruction now gives rise to a richer logic where additional connectives speaking about validity can be decomposed directly via their left and right rules, such as the *strong implication* $A \Rightarrow B$ which was previously ad hoc.

Note that we cannot easily model $\Diamond A$, which is not a *normal* modality in the technical sense that it does *not* satisfy $\Diamond(A \multimap B) \multimap (\Diamond A \multimap \Diamond B)$. Reed [25] provides a less direct, but adequate encoding that we elide here. Licata et al. [16] use the 2-categorical structure that generalizes our preorder to provide a more elegant representation.

▶ **Theorem 14.** *If we let τ embed propositions of normal judgmental S4 into the instance of adjoint logic described above, then*
1. $\Delta; \Gamma \vdash A$ *in judgmental S4 iff* ${\uparrow}_{\mathsf{U}}^{\mathsf{V}} \tau(\Delta), \tau(\Gamma) \vdash_I \tau(A)$.
2. $\Delta; \cdot \vdash A$ *in judgmental S4 iff* ${\uparrow}_{\mathsf{U}}^{\mathsf{V}} \tau(\Delta) \vdash_I {\uparrow}_{\mathsf{U}}^{\mathsf{V}} \tau(A)$.

▶ **Example 15** (Lax logic). Lax logic [8, 23] encodes a weaker form of truth called *lax truth*. We can represent it as a substructural adjoint logic with two modes, $\mathsf{U} > \mathsf{X}$, where both modes satisfy weakening and contraction. We restrict the lax layer to a single connective and omit additive connectives for simplicity.

$$\begin{array}{lcl} A_{\mathsf{U}}, B_{\mathsf{U}} & ::= & p_{\mathsf{U}} \mid A_{\mathsf{U}} \rightarrow B_{\mathsf{U}} \mid A_{\mathsf{U}} \times B_{\mathsf{U}} \mid 1_{\mathsf{U}} \mid {\uparrow}_{\mathsf{X}}^{\mathsf{U}} A_{\mathsf{X}} \\ A_{\mathsf{X}} & ::= & {\downarrow}_{\mathsf{X}}^{\mathsf{U}} A_{\mathsf{U}} \end{array}$$

Now the lax modality is defined as $\bigcirc A_{\mathsf{U}} = {\uparrow}_{\mathsf{X}}^{\mathsf{U}} {\downarrow}_{\mathsf{X}}^{\mathsf{U}} A_{\mathsf{U}}$.

We can now add further connectives directly operating on the lax layer and obtain consistent left and right rules for them.

▶ **Theorem 16.** *If we let τ embed propositions of lax logic into the instance of adjoint logic described above, then*

1. $\Gamma \vdash A$ *true in lax logic iff* $\tau(\Gamma) \vdash_I \tau(A)$.
2. $\Gamma, \Gamma' \vdash A$ *lax in lax logic iff* $\tau(\Gamma), {\downarrow}_{\mathsf{X}}^{\mathsf{U}}\tau(\Gamma') \vdash_I {\downarrow}_{\mathsf{X}}^{\mathsf{U}}\tau(A)$.

▶ **Example 17** (Intuitionistic Subexponential Linear Logic)**.** We can represent a somewhat restricted form of intuitionistic subexponential linear logic (ISELL) [5] as a fragment of adjoint logic. Subexponential labels of *zones* correspond to modes, and we preserve the preorder between labels as the preorder between modes. There is a *working zone* which corresponds to a distinguished mode L.

We require $z \geq \mathsf{L}$ for all modes $z \neq \mathsf{L}$ and define $!_z A = {\downarrow}_{\mathsf{L}}^{z} {\uparrow}_{\mathsf{L}}^{z} A$ for $z > \mathsf{L}$. We also work on the ?-free fragment (for much the same reasons that we work with the normal fragment of judgmental S4 in Example 13), making this slightly less general than ISELL, which also includes $?_z A$ and allows labels $z < \mathsf{L}$. Indeed, the rules for the shifts under the obvious candidate representation $?_z A = {\uparrow}_{z}^{\mathsf{L}} {\downarrow}_{z}^{\mathsf{L}} A$ do not match the rules for $?_z A$ in ISELL. Fortunately, the modality $?_z$ is not in the image of the translation [5, Section 4.1] from classical subexponential logic [19] into ISELL, so it does not appear essential to gauge its expressive power.

An instance of ISELL satisfying these requirements can then be seen as an instance of adjoint logic where all modes a other than L contain only propositions of the form ${\uparrow}_{\mathsf{L}}^{a} A_{\mathsf{L}}$.

We also have a new opportunity, namely adding connectives that directly combine propositions of mode $z \neq \mathsf{L}$. These additional connectives may reduce the number of subexponential modalities in a logic representation. This in turn streamlines the focusing behavior of encodings since subexponential modalities interrupt focusing phases. For example, the encodings proposed by Nigam et al. [20] for minimal logic (G1m in their Figure 4) or for lax logic (in their Figure 12) require many modalities which interrupt focusing phases in the represented logic, even while focusing is present in the metalogic. No such indirections are needed here.

Since the relevant fragments of ISELL are also a fragment of ADJ^F (see Section 4), some representations undertaken by Chaudhuri [5] (for classical logics, for example) also provide additional examples for ADJ^F, but we have not yet investigated whether the increase in expressiveness can be exploited for further results. Note that Chaudhuri only considers linear modalities and those with both weakening and contraction, so we slightly generalize his version of ISELL along this dimension.

▶ **Theorem 18.** *If we let τ embed propositions of an instance of ISELL into the corresponding instance of adjoint logic as described above, then*

1. $\Delta, !_{a_1} A_1, \ldots, !_{a_n} A_n \vdash B$ *in ISELL iff* $\tau(\Delta), {\uparrow}_{\mathsf{L}}^{a_1}\tau(A_1), \ldots, {\uparrow}_{\mathsf{L}}^{a_n}\tau(A_n) \vdash_E \tau(B)$.
2. $!_{a_1} A_1, \ldots, !_{a_n} A_n \vdash !_b B$ *in ISELL iff* ${\uparrow}_{\mathsf{L}}^{a_1}\tau(A_1), \ldots, {\uparrow}_{\mathsf{L}}^{a_n}\tau(A_n) \vdash_E {\uparrow}_{\mathsf{L}}^{b}\tau(B)$.

We use $\vdash_E$ rather than $\vdash_I$ here because ISELL's explicit structural rules make it more direct to embed ISELL into ADJ^E rather than ADJ^I.

4 Focused Adjoint Logic

We begin by polarizing the propositions of ADJ, giving us the following syntax for propositions:

$$\begin{array}{llll} \textit{Negative propositions} & A_m^-, B_m^- & ::= & p_m^- \mid A_m^+ \multimap_m B_m^- \mid \&_{j\in J} {A_m^j}^- \mid {\uparrow}_k^m A_k^+ \\ \textit{Positive propositions} & A_m^+, B_m^+ & ::= & p_m^+ \mid A_m^+ \otimes_m B_m^+ \mid \mathbf{1}_m^+ \mid \oplus_{j\in J} {A_m^j}^+ \mid {\downarrow}_m^\ell A_\ell^- \end{array}$$

Here, p_m^- and p_m^- are negative and positive atoms, respectively.

We have chosen all shifts to reverse polarity: $({\downarrow}_k^m A_m^-)^+$ and $({\uparrow}_k^m A_k^+)^-$. We believe two additional polarizations of shift operators are possible, namely $({\downarrow}_k^m A_m^+)^+$ and $({\uparrow}_m^k A_m^-)^-$. These would appear as "regular" positive or negative logical operators and might streamline some encodings. When we de-polarize they will, of course, be indistinguishable from their polarity-reversing cousins. We leave more detailed investigation of these additional polarity-preserving shifts to future work.

We then present our focused system in the style used in [26], using the following grammar for the components of our sequents:

$$\begin{array}{llll} \textit{Stable antecedents} & \Psi & ::= & \cdot \mid A_m^- \mid \langle A_m^+ \rangle \mid \Psi, \Psi' \\ \textit{Inversion antecedents} & \Omega & ::= & \cdot \mid A_m^+ \bullet \Omega \\ \textit{Succedents} & U & ::= & [A_m^+] \mid A_m^+ \mid A_m^- \mid \langle A_m^- \rangle \\ \textit{Ordered antecedents} & L & ::= & \Omega \mid [A_m^-] \end{array}$$

We use $\bullet$ rather than , to separate propositions in the (ordered) inversion context Ω to emphasize that those contexts are to be treated as lists rather than as multisets.

Similarly to the stable antecedents, only the succedents A_m^+ and $\langle A_m^- \rangle$ are stable (as A_m^- is invertible on the right, and $[A_m^+]$ is in focus). We will use stability of succedents U as a side condition of some proofs (as stability of the antecedent is implied by having an empty inversion context Ω).

Propositions in square brackets ($[A_m^+]$ or $[A_m^-]$) are propositions in focus, while propositions in angle brackets ($\langle A_m^+ \rangle$ or $\langle A_m^- \rangle$) are suspended propositions. In the system of Figure 3, suspended atoms $\langle p_m^+ \rangle$ and $\langle p_m^- \rangle$ are used so they can appear in stable sequents as antecedents with otherwise positive and in succedents with otherwise negative propositions, respectively. Suspending arbitrary propositions is a technical device introduced by Simmons [26] that allows for a structural proof of identity expansion (see Section 4.1).

With these parts, we have the following three types of sequents:

$$\begin{array}{lll} \textit{Right focus} & \Psi \vdash_F [A_m^+] & \\ \textit{Inversion} & \Psi \,;\, \Omega \vdash_F U & (\text{where } U \neq [A_m^+]) \\ \textit{Left focus} & \Psi \,;\, [A_m^-] \vdash_F U & (\text{where } U \text{ is } \textit{stable}) \end{array}$$

Each of these sequents is a special case of the general form $\Psi \,;\, L \vdash_F U$, but it is useful to separate these cases for some theorem statements and proofs. The constraints on what form U may take in each sequent are standard for intuitionistic focused systems [14, 26], and serve to ensure that at most one formula is in focus at a time, and that if a formula is in focus, then there are no formulae in inversion.

Most of the rules of ADJ^F arise straightfowardly from their ADJ^I counterparts by having the principal formula either in focus or in inversion (depending on its polarity and whether it is on the left or the right). We also add the focus^+ and focus^- rules, which allow us to focus on positive propositions on the right and negative propositions on the left, as well as the susp^+ and susp^- rules, which allow us to suspend atomic propositions (although we will later prove that versions of these rules which suspend arbitrary propositions are admissible in

Theorem 24). The identity rule of ADJ^I is split into the id^+ and id^- rules as in all focusing systems [1, 14].

We now work to prove defocalization and focalization results which show that ADJ^I and ADJ^F are equivalent. Composing these results with the results of Section 3, we will get that ADJ^E and ADJ^F are equivalent.

In order to state (and prove) our defocalization theorem, we need to define an erasure operation taking propositions and contexts in ADJ^F to propositions and contexts in ADJ^I.

▶ **Definition 19** (Erasure). Given a context Ψ, antecedent L, or succedent U, we define the *erasure* of Ψ, L, or U, denoted by $(\Psi)^\bullet$ or $(U)^\bullet$, to be the result of removing all focusing and suspension brackets. Formally, we give the following inductive definition:

$$
\begin{array}{rcl}
(\cdot)^\bullet & = & (\cdot) \\
(A_m^-, \Psi)^\bullet & = & A_m, (\Psi)^\bullet \\
(\langle A_m^+ \rangle, \Psi)^\bullet & = & A_m, (\Psi)^\bullet
\end{array}
\qquad
\begin{array}{rcl}
([A_m^-])^\bullet & = & A_m \\
(\cdot)^\bullet & = & (\cdot) \\
(A_m^+ \bullet \Omega)^\bullet & = & A_m, (\Omega)^\bullet
\end{array}
\qquad
\begin{array}{rcl}
(\langle A_m^- \rangle)^\bullet & = & A_m \\
([A_m^+])^\bullet & = & A_m \\
(A_m^-)^\bullet & = & A_m \\
(A_m^+)^\bullet & = & A_m
\end{array}
$$

▶ **Theorem 20** (Defocalization). *If $\Psi \,;\, L \vdash_F U$, then $(\Psi)^\bullet, (L)^\bullet \vdash_I (U)^\bullet$.*

Proof. We prove this by noting that each (erased) rule of the focused system is either a rule of ADJ^I or a no-op (as in the case of the $\mathsf{focus}^\pm$ or $\mathsf{susp}^\pm$ rules). As such, we translate the ADJ^F proof into an ADJ^I proof rule-by-rule, removing the no-op rules. ◀

4.1 Focalization

Our path to proving focalization is much the same as that used in [26], relying on admissibility of cut and admissibility of suspension for general propositions. In order to prove cut admissible, we must first prove admissibility of weakening and contraction (Lemma 21). These follow much the same pattern as the comparable lemmas for ADJ^I.

▶ **Lemma 21** (Admissibility of weakening and contraction for ADJ^F). *Take P_m to be either A_m^- or $\langle A_m^+ \rangle$.*

1. *If $\Psi \,;\, L \vdash_F U$ and $\mathsf{W} \in \sigma(m)$, then $\Psi, P_m \,;\, L \vdash_F U$.*
2. *If $\Psi, P_m, P_m \,;\, L \vdash_F U$ and $\mathsf{W} \in \sigma(m)$, then $\Psi, P_m \,;\, L \vdash_F U$.*

Proof. This follows from inductions over the proof that $\Psi \,;\, L \vdash_F U$ and the proof that $\Psi, P_m, P_m \,;\, L \vdash_F U$. ◀

The four cases of our cut admissibility theorem (Theorem 22) correspond to, respectively, positive principal cuts, negative principal cuts, left-commutative cuts, and right-commutative cuts.

▶ **Theorem 22** (Cut admissibility for ADJ^F). *Assuming $\mathsf{C} \in \sigma(\Psi_2)$, $\Psi_1, \Psi_2 \geq m$, and that Ψ_1, Ψ_2, Ψ_3, and U contain no non-atomic suspended propositions:*

1. *If $\Psi_1, \Psi_2 \vdash_F [A_m^+]$ and $\Psi_2, \Psi_3 \,;\, A_m^+ \bullet \Omega \vdash_F U$, then $\Psi_1, \Psi_2, \Psi_3 \,;\, \Omega \vdash_F U$.*
2. *If $\Psi_1, \Psi_2 \,;\, \cdot \vdash_F A_m^-$ and $\Psi_2, \Psi_3 \,;\, [A_m^-] \vdash_F U$ and U stable, then $\Psi_1, \Psi_2, \Psi_3 \,;\, \cdot \vdash_F U$.*
3. *If $\Psi_1, \Psi_2 \,;\, L \vdash_F A_m^+$ and $\Psi_2, \Psi_3 \,;\, A_m^+ \vdash_F U$ and U stable, then $\Psi_1, \Psi_2, \Psi_3 \,;\, L \vdash_F U$.*
4. *If $\Psi_1, \Psi_2 \,;\, \cdot \vdash_F A_m^-$ and $\Psi_2, \Psi_3, A_m^- \,;\, L \vdash_F U$ and U stable, then $\Psi_1, \Psi_2, \Psi_3 \,;\, L \vdash_F U$.*

Proof. This proceeds in a relatively standard nested induction, except that cases (3) and (4) depend on cases (1) and (2), respectively. As such, we prove this by induction over the (lexicographically ordered) quadruple $(A_m^\pm, i, \mathcal{D}, \mathcal{E})$, where $A_m^\pm$ is the formula cut out, i is

$$\frac{\mathsf{W} \in \sigma(\Psi)}{\Psi, \langle p_m^+ \rangle \vdash [p_m^+]}\ \mathsf{id}^+$$

$$\frac{\Psi_1 \geq m \quad \mathsf{W} \in \sigma(\Psi_2) \quad \Psi_1 \,;\, \cdot \vdash A_m^-}{\Psi_1, \Psi_2 \vdash [{\downarrow_k^m} A_m^-]}\ {\downarrow}R \qquad \frac{i \in J \quad \Psi \vdash \left[{A_m^i}^+\right]}{\Psi \vdash \left[\oplus_{j \in J} {A_m^j}^+\right]}\ \oplus R^i$$

$$\frac{\mathsf{C} \in \sigma(\Psi_2) \quad \Psi_1, \Psi_2 \vdash [A_m^+] \quad \Psi_2, \Psi_3 \vdash [B_m^+]}{\Psi_1, \Psi_2, \Psi_3 \vdash [A_m^+ \otimes B_m^+]}\ \otimes R \qquad \frac{\mathsf{W} \in \sigma(\Psi)}{\Psi \vdash [\mathbf{1}_m^+]}\ \mathbf{1}R$$

$$\frac{\Psi \vdash [A_m^+]}{\Psi \,;\, \cdot \vdash A_m^+}\ \mathsf{focus}^+ \qquad \frac{U \text{ stable} \quad \Psi, (A_m^-)^? \,;\, [A_m^-] \vdash U}{\Psi, A_m^- \,;\, \cdot \vdash U}\ \mathsf{focus}^-$$

$$\frac{\Psi, \langle p^+ \rangle \,;\, \Omega \vdash U}{\Psi \,;\, p^+ \bullet \Omega \vdash U}\ \mathsf{susp}^+ \qquad \frac{\Psi \,;\, \cdot \vdash \langle p^- \rangle}{\Psi \,;\, \cdot \vdash p^-}\ \mathsf{susp}^-$$

$$\frac{\Psi, A_m^- \,;\, \Omega \vdash U}{\Psi \,;\, {\downarrow_k^m} A_m^- \bullet \Omega \vdash U}\ {\downarrow}L \qquad \frac{\Psi \,;\, {A_m^j}^+ \bullet \Omega \vdash U \quad \text{for all } j \in J}{\Psi \,;\, \oplus_{j \in J} {A_m^j}^+ \bullet \Omega \vdash U}\ \oplus L$$

$$\frac{\Psi \,;\, A_m^+ \bullet B_m^+ \bullet \Omega \vdash U}{\Psi \,;\, A_m^+ \otimes B_m^+ \bullet \Omega \vdash U}\ \otimes L \qquad \frac{\Psi \,;\, \Omega \vdash U}{\Psi \,;\, \mathbf{1}_m^+ \bullet \Omega \vdash U}\ \mathbf{1}L$$

$$\frac{\Psi \,;\, \cdot \vdash A_k^+}{\Psi \,;\, \cdot \vdash {\uparrow_k^m} A_k^+}\ {\uparrow}R \qquad \frac{\Psi \,;\, \cdot \vdash {A_m^j}^- \quad \text{for all } j \in J}{\Psi \,;\, \cdot \vdash \&_{j \in J} {A_m^j}^-}\ \& R$$

$$\frac{\Psi \,;\, A_m^+ \vdash B_m^-}{\Psi \,;\, \cdot \vdash A_m^+ \multimap B_m^-}\ {\multimap}R$$

$$\frac{\mathsf{W} \in \sigma(\Psi)}{\Psi \,;\, [p_m^-] \vdash \langle p_m^- \rangle}\ \mathsf{id}^-$$

$$\frac{k \geq \ell \quad \Psi \,;\, A_k^+ \vdash U_\ell}{\Psi \,;\, [{\uparrow_k^m} A_k^+] \vdash U_\ell}\ {\uparrow}L \qquad \frac{i \in J \quad \Psi \,;\, \left[{A_m^i}^-\right] \vdash U}{\Psi \,;\, \left[\&_{j \in J} {A_m^j}^-\right] \vdash U}\ \& L^i$$

$$\frac{\Psi_1, \Psi_2 \geq m \quad \mathsf{C} \in \sigma(\Psi_2) \quad \Psi_1, \Psi_2 \vdash [A_m^+] \quad \Psi_2, \Psi_3 \,;\, [B_m^-] \vdash U}{\Psi_1, \Psi_2, \Psi_3 \,;\, [A_m^+ \multimap B_m^-] \vdash U}\ {\multimap}L$$

Figure 3 Focused Adjoint Logic (ADJ^F).
"U stable" means that U is either A_m^+ or $\langle p_m^- \rangle$.

the case number in the theorem statement, $\mathcal{D}$ is the left-hand proof of the cut, and $\mathcal{E}$ is the right-hand proof of the cut.

Admissibility of weakening is key for the cases involving $\mathcal{D}$ or $\mathcal{E}$ being a leaf of the proof—either an $\mathsf{id}^{\pm}$ rule or a $\mathbf{1}R$ rule, while admissibility of contraction is used throughout the proof, primarily to handle propositions in Ψ_2. ◀

Now, we proceed to proving identity expansion. We wish to prove that if $\mathsf{W} \in \sigma(\Psi)$, then $\Psi \,;\, A_m^+ \vdash_F A_m^+$ and $\Psi, A_m^- \,;\, \cdot \vdash_F A_m^-$. As described in [26], however, this does not follow from induction on the structure of $A_m^{\pm}$, and so we take Simmons' approach of generalizing the identity and suspension rules. We first generalize the identity rules to allow arbitrary propositions, rather than only atomic propositions. It is easy to see that these more general identity rules are admissible, as the only rules (read bottom-up) which introduce suspended propositions are the $\mathsf{susp}^{\pm}$ rules, which only introduce atomic suspended propositions. As such, the only propositions to which identity can be applied are atomic. Despite this, the more general rules are important to the proof of identity expansion. As a preliminary to identity expansion, we prove two cut-like rules, referred to in [26] as *focal substitution*, admissible.

▶ **Theorem 23** (Focal substitution). *Assume $\Psi_1, \Psi_2 \geq m$ and $\mathsf{C} \in \sigma(\Psi_2)$.*
1. *If $\Psi_1, \Psi_2 \vdash_F [A_m^+]$ and $\Psi_2, \Psi_3, \langle A_m^+ \rangle \,;\, L \vdash_F U$, then $\Psi_1, \Psi_2, \Psi_3 \,;\, L \vdash_F U$.*
2. *If $\Psi_1, \Psi_2 \,;\, L \vdash_F \langle A_m^- \rangle$ and $\Psi_2, \Psi_3 \,;\, [A_m^-] \vdash_F U$, then $\Psi_1, \Psi_2, \Psi_3 \,;\, L \vdash_F U$.*

Proof. (1) follows by induction over the proof that $\Psi_2, \Psi_3, \langle A_m^+ \rangle \,;\, L \vdash_F U$, while (2) follows by induction over the proof that $\Psi_1, \Psi_2 \,;\, L \vdash_F \langle A_m^- \rangle$, using admissibility of weakening and contraction in a similar manner to the proof of admissibility of cut. ◀

▶ **Theorem 24** (Suspension Expansion for ADJ^F).
1. *If $\Psi, \langle A_m^+ \rangle \,;\, \Omega \vdash_F U$, then $\Psi \,;\, A_m^+, \Omega \vdash_F U$.*
2. *If $\Psi \,;\, \cdot \vdash_F \langle A_m^- \rangle$, then $\Psi \,;\, \cdot \vdash_F A_m^-$.*

Proof. This proof proceeds by induction on the structure of $A_m^{\pm}$, using focal substitution in each (non-atomic) case to eliminate the suspended proposition. ◀

It follows almost immediately from Theorem 24 that identity expansion in its standard form holds—we arrive at the desired identity proofs by applying the admissible general suspension rule, followed by a focus rule, followed by an identity rule.

▶ **Corollary 25** (Identity Expansion for ADJ^F).
1. *If $\mathsf{W} \in \sigma(\Psi)$, then $\Psi \,;\, A_m^+ \vdash_F A_m^+$.*
2. *If $\mathsf{W} \in \sigma(\Psi)$, then $\Psi, A_m^- \,;\, \cdot \vdash_F A_m^-$.*

With cut admissibility and identity expansion, we can now move on to prove focalization. Since we are concerned with stable sequents, when translating a proof in ADJ^I into a proof in ADJ^F, we stabilize the sequent by inserting shifts. Since A_m is logically equivalent (in ADJ^I) to $\uparrow_m^m A_m$, and similarly, A_m is logically equivalent to $\downarrow_m^m A_m$ (Example 26), the way in which we do this stabilization is irrelevant—it only matters that there is some way of doing so.

▶ **Example 26.** Given A_m, we can construct the following two proofs:

$$\dfrac{\dfrac{\mathsf{W} \in \sigma(\cdot)}{A_m \vdash_I A_m}\,\mathsf{id}}{A_m \vdash_I \uparrow_m^m A_m}\,{\uparrow}R \qquad \dfrac{m \geq m \quad \dfrac{\mathsf{W} \in \sigma(\cdot)}{A_m \vdash_I A_m}\,\mathsf{id}}{\uparrow_m^m A_m \vdash_I A_m}\,{\uparrow}L$$

▶ **Theorem 27** (Focalization). *If* $(\Psi^-)^\bullet \vdash_I (A_m^+)^\bullet$*, then* $\Psi^- \,;\, \cdot \vdash_F A_m^+$

Proof. We first note that cut elimination for ADJ^I allows us to only consider cut-free proofs.

We then proceed by induction over the proof of $(\Psi^-)^\bullet \vdash_I (A_m^+)^\bullet$.

In each case other than the identity cases, we apply the inductive hypothesis to the premise(s) of the last rule used, and then cut the result of this together with the result of applying that rule to identies.

For instance, if the last rule used was $\downarrow R$, then we have as our initial proof that

$$\dfrac{\Psi_1 \geq m \quad \mathsf{W} \in \sigma(\Psi_2) \quad \overset{\mathcal{D}}{(\Psi_1)^\bullet \vdash_I A_m}}{(\Psi_1)^\bullet, (\Psi_2)^\bullet \vdash_I {\downarrow_k^m} A_m}\ {\downarrow}R$$

From this, we can construct the following proof:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{(\Psi_1)^\bullet \vdash_I A_m}}{\Psi_1 \,;\, \cdot \vdash_F A_m^-}\ \text{i.h.}(\mathcal{D}) \qquad \dfrac{\dfrac{A_m^- \geq m \quad \mathsf{W} \in \sigma(\Psi_2) \quad \dfrac{\mathsf{W} \in \sigma(\cdot)}{A_m^- \,;\, \cdot \vdash_F A_m^-}\ \mathsf{id}}{\Psi_2, A_m^- \vdash_F [{\downarrow_k^m} A_m^-]}\ {\downarrow}R}{\Psi_2, A_m^- \vdash_F {\downarrow_k^m} A_m^-}\ \mathsf{focus}^+}{\Psi_1, \Psi_2 \,;\, \cdot \vdash_F {\downarrow_k^m} A_m^-}\ \mathsf{cut}$$

using the admissibility of cut and of identity to make the dotted inferences. ◀

5 Conclusion

We have developed adjoint logic as a generic way to combine multiple logics with varying structural properties. A particular instance of adjoint logic is given by a preorder between modes of truth, each of which may optionally satisfy weakening, contraction, or both. On one hand, we decompose the subexponentials of subexponential linear logic [19, 5, 20] into shifts that embed logics into each other, rather than coding all of them into a "master" logic. This retains syntax and semantics of the individual components, with the adjunction properties guaranteeing conservativity. On the other hand, we generalize Benton's LNL allowing many substructural (and structural) logics to be harmoniously combined, as can be seen from our examples. In ongoing work [24] we are investigating an operational interpretation of adjoint logic propositions as a rich language of session types, supporting new communication patterns such as multicast or distributed garbage collection by virtue of the underlying adjoint logic.

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