CHAPTER 9

Absolute Neighborhood Retracts and Shape Theory

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Absolute neighborhood retracts (ANR's) and spaces having the homotopy type of ANR's, like polyhedra and CW-complexes, form the natural environment for homotopy theory. Homotopy-like properties of more general spaces (shape properties) are studied in shape theory. This is done by approximating arbitrary spaces by ANR's. More precisely, one replaces spaces by suitable systems of ANR's and one develops a homotopy theory of systems. This approach links the theory of retracts to the theory of shape. It is, therefore, natural to consider the history of both of these areas of topology in one article. A further justification for this is the circumstance that both theories owe their fundamental ideas to one mathematician, Karol Borsuk. We found it convenient to organize the article in two sections, devoted to retracts and to shape, respectively.

1. Theory of retracts

The problem of extending a continuous mapping $f : A \to Y$ from a closed subset A of a space X to all of X, or at least to some neighborhood U of A in X, is very often encountered in topology. Karol Borsuk realized that the particular case, when $Y = X$ and f is the inclusion $i : A \to X$, deserves special attention. In this case, any extension of i is called a *retraction* (*neighborhood retraction*). If retractions exist, A is called a *retract* (*neighborhood retract*) of X. In his Ph.D. thesis "O retrakcjach i zbiorach związanych" ("On retractions and related sets"), defended in 1930 at the University of Warsaw, Borsuk introduced and studied these basic notions as well as the topologically invariant notion of *absolute retract* (abbreviated as AR). He thus laid the foundations of the *theory of retracts*. The very suggestive term *retract* was proposed by Stefan Mazurkiewicz (1888–1945), who was Borsuk's Ph.D. supervisor. The term *absolute retract* was suggested by Borsuk's colleague Nachman Aronszajn, also a student of Mazurkiewicz.

It appears that the original of Borsuk's thesis has been lost in the turmoils of the Second World War. However, its main results were published in [27]. *Absolute neighborhood retracts* (abbreviated as ANR) were introduced in [28]. In the beginning Borsuk only considered separable metric spaces, especially metric compacta. Other early contributions to

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K. Borsuk was born in Warsaw in 1905 and studied mathematics at the University of Warsaw. He spent the period 1931–1932 on postdoctoral studies with leading European topologists of that time (Karl Menger in Wien, Heinz Hopf in Zürich and Leopold Vietoris in Innsbruck). His habilitation at the University of Warsaw took place in 1934 and he became Professor in 1946. After retirement in 1975, he continued with activities at the Mathematical Institute of the Polish Academy of Science. On four different occasions, Borsuk spent a Winter semester in US (Princeton, Berkeley, Madison, New Brunswick), which contributed to the quick spreading of his theory of retracts and later, the theory of shape. Borsuk died in Warsaw in 1982. N. Aronszajn was born in 1907 in Warsaw, where he went to school and university, obtaining his Ph.D. in 1930. He then worked in Paris and Cambridge until 1948, when he emmigrated to US. There he spent most of his career at the University of Kansas in Lawrence. He died in Corvallis, Oregon in 1980.

the theory of retracts, due to K. Kuratowski [154] and R.H. Fox [107], also refer to these classes of spaces. Gradually, the theory was extended, first to arbitrary metric spaces by C.H. Dowker [79] and J. Dugundji [85], then to more general classes of spaces $\mathcal{C}$, closed under homeomorphic images and closed subsets, by S.-T. Hu [126], O. Hanner [121] and E.A. Michael [184].

An AR (ANR) for the class $\mathcal{C}$ is a space Y from $\mathcal{C}$, such that, whenever Y is a closed subset of a space X from $\mathcal{C}$, then Y is a retract (neighborhood retract) of X. A space Y is an *absolute extensor* (*absolute neighborhood extensor*) for the class $\mathcal{C}$, abbreviated as AE($\mathcal{C}$) (ANE($\mathcal{C}$)), provided, for every closed subset A of a space X from $\mathcal{C}$, every mapping $f : A \to Y$ extends to all of X (to some neighborhood U of A in X). It is not required that Y belongs to $\mathcal{C}$. Clearly, if Y is from $\mathcal{C}$ and is an absolute extensor for $\mathcal{C}$, then Y is also an absolute retract for $\mathcal{C}$. The terminology AE and ANE was introduced in [184]. Gradually it became clear that the class $\mathcal{C}$ of metric spaces gives the most satisfactory theory. Hence, if we speak of ANR's and do not specify $\mathcal{C}$, we mean ANR's for metric spaces. A rather detailed and reliable study of the spaces ANR($\mathcal{C}$) and ANE($\mathcal{C}$), for various classes $\mathcal{C}$, has been carried out in Hu's monograph [128].

Borsuk's work on the theory of retracts had its precedents. The most important among these is the *Tietze–Urysohn extension theorem.* It was first proved, for metric spaces by H. Tietze [225]. Then P.S. Uryson proved his famous lemma: If A and B are closed disjoint subsets of a normal space X, there exists a mapping $f : X \to I$ to the real line segment $I = [0, 1]$ such that $f|A = 0$ and $f|B = 1$ [232]. In the case of metric spaces, the assertion of *Urysohn's lemma* is an elementary fact, which was used in Tietze's argument. Replacing this fact by its generalization enabled Uryson to obtain the extension theorem

Ernest A. Michael, Professor at the University of Washington in Seattle, was born in Zürich in 1925. He obtained the Ph.D. in 1951 from the University of Chicago. Sze-Tsen Hu, Professor at the University of California in Los Angeles, was born in Huchow, China in 1914. He obtained the B.Sc. from the University of Nanking, China and the D.Sc. in 1959 from the University of Manchester.

Heinrich Tietze (1880–1964) was born in Schleinz, Austria. He studied in Wien, München and Göttingen and obtained his Ph.D. in 1904 in Wien, where he became Privatdozent in 1908. From 1910 to 1919 he was professor at the Technical University in Brno and it is during that period that he obtained his extension theorem. He spent the rest of his career at the universities of Erlangen and München. Pavel Samuilovich Uryson (1898–1924) was born in Odessa. He was a student of D.F. Egorov (1869–1931) and N.N. Luzin (1883–1950) in Moscow, where he obtained his Ph.D. in 1921. Urysohn was one of the most promising Russian mathematicians of his generation, when he lost his life at the age of 25 in a tragic accident, while swimming in the rough seas of French Bretagne. His collected papers fill up two volumes.

for normal spaces. In present terminology the theorem asserts that $I = [0, 1]$ and the real line $\mathbb{R}$ are AE's for normal spaces. Recently, J. Mioduszewski drew attention to the fact that the argument used by Uryson in constructing the mapping $f : X \to I$ appeared a year earlier (in a different context) in the only paper by W.S. Bogomolowa, a student of Luzin [25].

An important question raised in the early days of the theory of retracts was to determine whether an absolute retract Y for a class $\mathcal{C}$ is necessarily an absolute extensor for $\mathcal{C}$. This is true for many important classes $\mathcal{C}$. For separable metric spaces it was proved in [154] and for arbitrary metric spaces in [85]. To obtain this result, one first embeds Y in a normed vector space L, in such a way that it is a closed subset of its convex hull K. For L one can take the space of bounded mappings $f : Y \to \mathbb{R}$, which is even a Banach space [155, 248]. Then one applies the *Dugundji extension theorem* [85], an important generalization of the Tietze–Urysohn theorem. It asserts that every convex set in a normed vector space (more generally, in a locally convex vector space) is an absolute extensor for metric spaces. This result was made possible only after A.H. Stone proved that metric spaces are paracompact [223]. For separable metric spaces Dugundji's extension theorem was already known to Polish topologists. Note that paracompactness of these spaces is an elementary fact, because separable metric spaces are Lindelöf, hence, also paracompact. Dugundji's theorem was later generalized to *stratifiable spaces* [26], a class of spaces, introduced in [56], which includes both metric spaces and CW-complexes.

An important result in the theory of retracts was J.H.C. Whitehead's theorem that the adjunction space of a mapping $f : A \to Y$, where $A \subseteq X$, X and Y are compact ANR's, is again a compact ANR [244]. Another important result was obtained by Hanner. He considered *local* ANE's, i.e. spaces which admit an open covering formed by ANE's, and proved that for metric (more generally, for paracompact) spaces, every local ANE is an ANE [121]. This theorem implies, e.g., that (metric) manifolds are ANR's.

In introducing (compact) ANR's Borsuk wanted to generalize compact polyhedra in a way which excludes the pathology often present in arbitrary metric compacta. For example, compact AR's have the fixed-point property [27], but there exist acyclic (locally connected) continua in $\mathbb{R}^3$ which do not have this property [29]. Generalizing a sum theorem from [9], Borsuk proved that the union $X = A_1 \cup A_2$ of two compact ANR's is an ANR, provided $A_1 \cap A_2$ is an ANR [28]. This implies that every compact polyhedron is indeed an ANR. In the same paper he showed that in the class of finite-dimensional compacta, ANR's are characterized by *local contractibility*. For an infinite-dimensional compactum X, local contractibility alone is not sufficient to ensure that X be an ANR [31].

Kazimierz Kuratowski (1896–1980), one of the founders of the Polish topology school, was born and died in Warsaw. He obtained his Ph.D. from the University of Warsaw in 1921. He first worked at the Technical University in Lwów. Since 1934 he was Professor at the University of Warsaw. During the Nazi occupation of Poland, both Kuratowski and Borsuk lectured at the underground university in Warsaw. Clifford Hugh Dowker was born in 1912 in a rural area of Western Ontario. He obtained his B.A. and M.A. in Canada and his Ph.D. in 1938 in Princeton, where he came to study under Solomon Lefschetz (1884–1972). In 1950, during the period of McCarthyism, Dowker moved to England and eventually became Professor at Birckbeck College in London, where he worked until his retirement in 1979. He died in London in 1982. James Dugundji (1919–1985) was born in New York in a family of Greek immigrants. He obtained his B.A. degree from New York University in 1940. The same year he started his graduate studies at the University of North Carolina at Chapel Hill as a student of Witold Hurewicz (1904–1956). After spending four years of war in the US. Air Force, in 1946 he entered the Massachusetts Institute of Technology, where Hurewicz became Professor in 1945. Under him Dugundji obtained his Ph.D. in 1948. The same year he started teaching at the University of Southern California in Los Angeles, where he became Professor in 1958. The Swedish topologist Olof Hanner was born in Stockholm in 1922 and obtained his Ph.D. from the University of Stockholm in 1952 with a thesis which consisted of three of his papers on ANR's. He became interested in ANR's during a visit to the Institute for Advanced Study in Princeton in 1949/50, where he came in touch with the work of Ralph Hartzler Fox (1913–1973) and Lefschetz.

The important property LC^n (*local connectedness up to dimension* n) was introduced in 1930 in Lefschetz's book [158] (see p. 91) and studied further in [159]. Generalizing Borsuk's work, Kuratowski proved that an n-dimensional separable metric space X is an ANR if and only if it is LC^n [154]. The proof uses the fundamental concepts of *nerve of an open covering* and *canonical mapping*, whose origins can be traced back to the work of Alexandroff [2, 3] and Kuratowski [153], respectively. The Kuratowski theorem was later generalized to arbitrary metric spaces by several authors [151, 145, 87].

In 1973 W.E. Haver proved that a locally contractible metric space X, which is the union of a countable collection of finite-dimensional compacta, is an ANR [123]. This result had important consequences in the study of the space $\mathrm{PLH}(M)$ of piecewise linear homeomorphisms of a compact PL-manifold M. A.V. Chernavskiĭ proved in 1969 that the space $\mathrm{H}(M)$ of homeomorphisms of a compact manifold M is locally contractible [68]. A simplified proof of Chernavskiĭ's result was obtained by R.D. Edwards and R.C. Kirby [104]. It follows from this proof that, for a compact PL-manifold M, the space $\mathrm{PLH}(M)$ is also locally contractible. On the other hand, R. Geoghegan showed that, for a compact polyhedron P, $\mathrm{PLH}(P)$ is the union of a countable collection of compact finite-dimensional sets [113]. Consequently, Haver's result applies and yields the conclusion that, for a compact PL-manifold M, $\mathrm{PLH}(M)$ is an ANR. For compact topological manifolds M, the question if $\mathrm{H}(M)$ is an ANR, is still open. The analogous question for Q-manifolds was answered in the affirmative, independently by S. Ferry [105] and by H. Toruńczyk [229].

In general, the geometric realization of an infinite *simplicial complex* K can be endowed with the *weak topology* (also called CW-*topology*) or the *metric topology* [161]. The resulting spaces will be denoted by $|K|$ and $|K|_m$, respectively. We refer to spaces $|K|$ as *polyhedra*. A polyhedron $|K|$ is metrizable if and only if the complex K is locally finite.

S. Lefschetz was born in Moscow and educated in Paris. After working for some years in industry, he turned to mathematics (following an industrial accident in which he lost both hands). In 1925 he joined the Mathematics Department in Princeton, where he became a leading topologist, together with O. Veblen (1880–1960) and J.W. Alexander (1888–1971). After retiring from Princeton University, he continued his activities at Brown University and in Mexico. John Henry Constantine Whitehead (1904–1960), another leading topologist, was born in India and educated in Oxford. He continued his studies in Princeton under Veblen and there obtained his Ph.D. in 1931. He became Professor in Oxford in 1945. He died in Princeton, where he was spending a year's leave.

In this case the weak topology and the metric topology coincide. Polyhedra are special cases of CW-*complexes* (CW-*spaces*), introduced by Whitehead in [246]. It was shown by Dugundji [86] that CW-complexes are paracompact spaces and ANE's for metric spaces. For polyhedra, the latter assertion was proved independently by Y. Kodama [144]. If a polyhedron $|K|$ is locally compact, then the complex K is locally finite and therefore, $|K| = |K|_m$ must be an ANR.

For an arbitrary simplicial complex K, the space $|K|_m$ is an ANR. To prove this important fact, one first proves the assertion in the special case of *full* simplicial complexes, i.e. complexes where every finite set of vertices spans a simplex. This is easily done by applying Dugundji's extension theorem. In the general case, one needs the fact that, for every subcomplex $L \subseteq K$, $|L|_m$ is a neighborhood retract of $|K|_m$. The standard argument consists of showing that, in the first barycentric subdivision K' of K, the star of the carrier of L' is an open set in the carrier of K', which retracts to the carrier of L'. However, to apply this argument, one needs to know that $|K|_m = |K'|_m$. This was proved by Lefschetz in [161], a monograph devoted entirely to local n-connectedness and retraction.

The French topologist Robert Cauty studied closely the relationship between polyhedra and CW-complexes. In particular, in [50] he characterized spaces which embed into polyhedra as closed subsets. All CW-complexes satisfy his criterion. Moreover, if a CW-complex X is embedded as a closed subset of a polyhedron P, then there exists an open neighborhood U of X in P which retracts to X. It is well known that every open subset of a polyhedron is itself a polyhedron. Therefore, CW-complexes are retracts of polyhedra. Twenty years later, Cauty showed that an open subset of a CW-complex need not be a CW-complex [53]. He thus corrected an error, appearing occasionally in the literature.

Cauty showed that there exist CW-complexes which are not ANR's for paracompact (hereditarily paracompact) spaces [49]. An example, due to E. van Douwen and R. Pol [78] shows that, there exist a regular countable space X (hence, a Lindelöf space), a closed subset $A \subseteq X$ and a mapping f of A to a 1-dimensional polyhedron $|K|$, which does not extend to any neighborhood of A. Consequently, $|K|$ is a not an ANE for paracompact spaces. On the other hand, for a simplicial complex K with no infinite simplices, $|K|_m$ is an ANE even for collectionwise normal spaces [52]. This shows that the extension properties of complexes depend essentially on the choice of the topology.

Cauty proved that every CW-complex is an ANR for stratifiable spaces [51]. This was achieved using *topological convexity* (abbreviated as TC) and *local topological convexity* (abbreviated as TLC). A space X is TLC provided there exists a neighborhood U of the diagonal Δ in $X \times X$ and there exists a mapping $\phi : U \times I \to X$ such that $\phi(x, y, 0) = x$,

R. Cauty was born in 1946. He studied in Paris and belonged to M. Zisman's Algebraic Topology Seminar. He obtained his doctorat d'état in 1972. Since in Paris there was not much interest in General Topology, Cauty learned the subject by himself, beginning with Kuratowski's *Topologie*. ANR's and complexes, being the meeting ground of General and Algebraic Topology, constituted the natural topic of his research.

$\phi(x, y, 1) = y$, for all $(x, y) \in U$ and $\phi(x, x, t) = x$, for all $x \in X$, $t \in I$. In addition, one requires that every point $x \in X$ admits a basis of neighborhoods V such that $V \times V \subseteq U$ and $\phi(V \times V \times I) \subseteq V$. Property TC is obtained by requiring that $U = X \times X$. Clearly, locally convex topological vector spaces have property TC. There exist compact ANR's which are not TLC-spaces [42] (also see [33], Ch. VI.4).

A weaker notion, called *equiconnectedness* (*local equiconnectedness*) was already considered by Fox [108] and J.-P. Serre [212], who used the abbreviations UC (ULC). These properties are obtained from properties TC (TLC) by omitting the additional condition $\phi(V \times V \times I) \subseteq V$. It is easy to see that every AR (ANR) is a UC-space (ULC-space). Finite-dimensional metric ULC-spaces are ANR's [87]. For infinite-dimensional metric ULC-spaces, one finds in [88, 125] additional conditions, which make these spaces ANR's. The question whether every metric ULC-space is an ANR, remained open for a long time. Only recently, a counterexample was obtained by Cauty, who exhibited a metric linear space (hence, a UC-space), which is not an AR [55]. Cauty's example depends essentially on the existence of dimension-raising cell-like mappings of compacta [83].

In the literature there are many results characterizing ANR's. Here we mention a classical criterion, based on realizations of simplicial complexes K with respect to a covering $\mathcal{U}$. A *full realization* of K is a mapping $g : |K| \to X$ of the geometric realization of K (CW-topology) such that every (closed) simplex $\sigma \in K$ maps into some member U of $\mathcal{U}$. A *partial realization* is a mapping $f : |L| \to X$, defined on the carrier of some subcomplex L of K such that, for every $\sigma \in K$, the set $f(|L| \cap \sigma)$ is contained in some member U of $\mathcal{U}$. A metric space X is an ANR if and only if every open covering $\mathcal{U}$ of X admits a refinement $\mathcal{V}$ such that, for every subcomplex $L \subseteq K$, which contains all the vertices of K, every partial realization $f : |L| \to X$ with respect to $\mathcal{V}$ admits an extension to a full realization $g : |K| \to X$ with respect to $\mathcal{U}$. This was proved in [160], for compact metric spaces and in [87], for arbitrary metric spaces. The problem of finding convenient characterizations of infinite-dimensional ANR's still deserves attention.

A very useful theorem on ANR's asserts that sufficiently near mappings into an ANR must be homotopic. More precisely, if $\mathcal{U}$ is an open covering of an ANR Y, then there exists an open covering $\mathcal{V}$ such that any two $\mathcal{V}$-near mappings $\phi, \psi : X \to Y$ are $\mathcal{U}$-homotopic, i.e. are connected by a homotopy $H : X \times I \to Y$ with paths $H(x \times I)$, $x \in X$, contained in members of $\mathcal{U}$ [85, 120]. ANR's can be characterized as metrizable spaces Y having the property that, for every open covering $\mathcal{U}$, Y is $\mathcal{U}$-homotopy dominated by some polyhedron P, i.e. there exist mappings $f : Y \to P$, $g : P \to Y$ such that gf and id are $\mathcal{U}$-homotopic [87, 120]. Necessity of the condition is a consequence of the fact that every covering $\mathcal{V}$ of an ANR Y admits a polyhedron P and admits mappings f, g such that gf and id are $\mathcal{V}$-near mappings. Spaces Y having this property are called *approximate polyhedra* [173]. That every ANR Y is an approximate polyhedron is a consequence of the *bridge theorem,* which asserts that, for every mapping $f : X \to Y$ of a space into an ANR and for every open covering $\mathcal{V}$ of Y, there exists a normal covering $\mathcal{U}$ of X and a mapping $g : |N(\mathcal{U})| \to Y$ of the geometric realization of the nerve $N(\mathcal{U})$, such that, for any canonical mapping $p : X \to |N(\mathcal{U})|$, the mappings f and gp are $\mathcal{V}$-near [127].

It was John Milnor who in 1959 renewed the interest of topologists in the class of spaces having the homotopy type of CW-complexes [187]. ANR's belong to this class, because every ANR X has the *homotopy type* of the *geometric realization* $|S(X)|$ of its singular complex $S(X)$. For an arbitrary space X, there is a canonical mapping $j_X : |S(X)| \to X$, which is a weak homotopy equivalence, i.e. it induces isomorphisms of homotopy groups. By a well-known theorem, a weak homotopy equivalence between CW-complexes is a homotopy equivalence [129, 245]. This theorem readily extends to spaces homotopy dominated by CW-complexes and, therefore, applies to j_X, whenever X is an ANR. For every space X, $|S(X)|$ is triangulable and thus, every ANR has the homotopy type of a polyhedron. Actually, the geometric realization $|K|$ of any simplicial set K is a polyhedron. The proof given in [13] and reproduced in [169] contained an error, which was corrected in the Ph.D. thesis of Rudolf Fritsch, a student of Dieter Puppe [110–112].

Conversely, every polyhedron P has the homotopy type of an ANR. Indeed, if K is a simplicial complex such that $P = |K|$, then the identity mapping $|K| \to |K|_m$ is a homotopy equivalence [80]. However, $|K|_m$ is an ANR. A recent result of Cauty characterizes ANR's as metric spaces all of whose open subsets have the homotopy types of ANR's [54].

One of the most useful results on ANR's is Borsuk's *homotopy extension theorem* [30]. A pair of spaces (X, A) is said to have the *homotopy extension property* (abbreviated as HEP) with respect to a space Y, provided every mapping $f : (X \times 0) \cup (A \times I) \to Y$ admits an extension $F : X \times I \to Y$. Borsuk's theorem asserts that every pair, where X is a metric space and A is closed, has HEP with respect to any ANR Y. Among many generalizations of this theorem, especially interesting was the result of Dowker, which asserts that pairs, where $X \times I$ is normal and A is closed, have HEP with respect to separable Čech complete ANR's, in particular, with respect to compact ANR's [81]. This result naturally led to the question, does normality of X imply normality of $X \times I$? This proved to be a very challenging problem, which generated much research in general topology. It was finally solved in the negative by Mary Ellen Rudin [206].

After the development of the theory of fibrations [212, 130], it became clear that, for a pair (X, A), HEP with respect to all spaces Y, viewed as a property of the inclusion $A \to X$, is a notion dual to the notion of fibration, hence, it is referred to as a *cofibration*. Cofibration pairs (X, A) are also called *neighborhood deformation pairs* and play an important role in homotopy theory.

One of the central problems of geometric topology in the last decades has been the *recognition problem* for topological manifolds: Find a list of topological properties which characterize manifolds among topological spaces. The properties should be easy to check, hence, they should not use notions like homeomorphisms. In 1978 James W. Cannon solved the famous *double suspension problem*, by showing that the double suspension of a homology 3-sphere is homeomorphic to the 5-sphere S^5 [47]. This work led him to state the following conjecture, which would solve the recognition problem. *Cannon's conjecture*: A topological space X is an n-manifold (separable metric), $n \geqslant 5$, if and only if it is a *homology n-manifold* having the *disjoint disc property*. By definition, homology n-manifolds are finite-dimensional separable locally compact ANR's X, whose local homology groups (integer coefficients) coincide with those of $\mathbb{R}^n$, i.e. $H_m(X, X \backslash \{x\}; \mathbb{Z}) \approx$

$H_m(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}; \mathbb{Z})$, for all m. A metric space (X, d) has the disjoint disc property provided, for any two mappings $f_1, f_2 : B^2 \to X$ of the 2-ball B^2 and any $\varepsilon > 0$, there exist two mappings $g_1, g_2 : B^2 \to X$, such that $d(f_i, g_i) < \varepsilon$, $i = 1, 2$, and the images $g_1(B^2)$ and $g_2(B^2)$ are disjoint. Note that ANR's play an important role in this conjecture.

A major step towards proving Cannon's conjecture was the *cell-like approximation theorem* of R.D. Edwards, which considerably strengthened earlier work on cell-like mappings between manifolds [214]. Edwards announced his result in 1977 and published an outline of the proof in [98]. A detailed proof, for $n \geqslant 6$, appeared in Daverman's monograph [77]. The Edwards theorem asserts that a cell-like mapping $f : M \to X$ from an n-manifold M to a finite-dimensional space X is a near-homeomorphism, i.e. can be approximated by homeomorphisms, if and only if it has the *disjoint disc property*. Consequently, Cannon's conjecture is true provided X is the image of an n-manifold M under a cell-like mapping f. Edwards theorem is the crown of years of efforts of many geometric topologists. An essential ingredient in the proof is R.H. Bing's *shrinking criterion*, which gives necessary and sufficient conditions in order that a proper mapping $f : X \to Y$ be approximable by homeomorphisms. The criterion requires that for every pair of open coverings $\mathcal{U}$ of X and $\mathcal{V}$ of Y, there exists a homeomorphism $h : X \to X$ having the following properties:

(i) The mappings $fh, f : X \to Y$ are $\mathcal{V}$-near.

(ii) For every $y \in Y$, there exists a U in $\mathcal{U}$ such that $h(f^{-1}(y)) \subseteq U$ [182].

In view of the cell-like approximation theorem, to complete the proof of Cannon's conjecture, it would have been sufficient to show that every homology n-manifold X, $n \geqslant 5$, is *resolvable*, i.e. it is the cell-like image of an n-manifold. Frank Quinn discovered an integer-valued obstruction $i(X) \equiv 1 (\mathrm{mod}\, 8)$ and showed that the above question has a positive answer if and only if $i(X) = 1$ [203, 204]. For a while it was not known if there actually exist homology manifolds with $i(X) \neq 1$. The existence of such homology manifolds is a major recent achievement in topology, due to J. Bryant, S. Ferry, W. Mio and S. Weinberger [44].

A mapping $f : X \to Y$ is *cell-like* provided it is proper (counter-images of compact sets are compact) and all the fibers $f^{-1}(y)$, $y \in Y$, are *cell-like spaces*, i.e. have the shape of a point. Cell-like spaces and mappings were studied before the advent of shape theory. Note that a space X is cell-like if and only if every mapping $f : X \to P$ to an ANR P is homotopic to a constant mapping. A metric space X is cell-like if and only if for every embedding in an ANR M the following UV^∞ *property* holds: For every neighborhood U of X in M, there exists a neighborhood V of X such that $V \subseteq U$ and the inclusion $i : V \to U$ is nullhomotopic. A special case of cell-likeness is *cellularity* of sets in an n-manifold M, a notion introduced by Morton Brown in connection with the Schoenflies problem [43]. A subset X of an n-manifold M is cellular in M if there exists a sequence (B_i^n) of n-dimensional balls in M such that $B_{i+1}^n \subseteq \operatorname{Int} B_i^n$, for all i, and $X = \bigcap_i B_i^n$. Cellularity was studied extensively by D.R. McMillan, Jr. [183]. For a survey on cell-like mappings see [157]. Mappings between ANR's with AR-fibers as well as mappings satisfying the corresponding LC^n and n-contractibility conditions were studied already in [217].

Problems encountered in the research concerning infinite-dimensional manifolds, especially manifolds modelled on the Hilbert space l_2 and the Hilbert cube Q, were similar to problems encountered in the research concerning n-manifolds and progress in one area often stimulated progress in the other one. In many cases the infinite-dimensional problems turned out to be more accessible than the corresponding finite-dimensional problems and

R.H. Bing (1914–1986) was a student of the legendary topology teacher Robert Lee Moore (1882–1974) at the University of Texas in Austin. Bing obtained his Ph.D. in Austin in 1945. He did pioneering work concerning decomposition spaces and homeomorphisms in 3-dimensional manifolds [22]. The first systematic study of homology manifolds is due to another student of Moore, Raymond Louis Wilder (1896–1982) [247].

the solution of the former preceded the solution of the latter. The center of this research was the group around Richard Davis Anderson, Professor at the University of Louisiana in Baton Rouge.

R.D. Anderson was born in Hamden, Connecticut in 1922. He was a student of R.L. Moore at Austin, Texas, where he obtained his Ph.D. in 1948. One can associate with the Anderson group T.A. Chapman, D.W. Curtis, S. Ferry, R. Geoghegan, D.W. Henderson, R.M. Schori, J.E. West, R.Y.T. Wong.

The direct product of an n-manifold by the Hilbert space l_2 is obviously an l_2-manifold. In 1960 V. Klee asked the converse. Is every l_2-manifold homeomorphic to the product of an n-manifold with l_2? In 1961 in a surprising article Borsuk answered this question in the negative [32]. He also posed the following intriguing problems: Is it true that the cartesian product of a compact polyhedron (ANR) by Q is a Q-manifold? Is it true that every Q-manifold is homeomorphic to the product of a compact polyhedron by Q?

A very special case of the first problem, contributed by Borsuk to the Scottish book in 1938, asked whether the product of a triod with Q is homeomorphic to Q. It was answered affirmatively by Anderson in 1964. The first problem for (locally compact) polyhedra was answered affirmatively in 1970 by West [240]. In 1973 Chapman developed a procedure to perform surgery on infinite-dimensional manifolds, which enabled him to establish an infinite-dimensional version of the handle-straightening theorem of R.C. Kirby and L.C. Siebenmann [143]. This result was an essential ingredient in the proof of two important theorems of Chapman. The first one was the *triangulation theorem*, which answered affirmatively the second of the Borsuk problems [64]. The second one was an unexpected proof of the topological invariance of the Whitehead torsion, i.e. proof of the assertion that homeomorphisms between compact polyhedra are simple homotopy equivalences. The solution of this more than 20 years old finite-dimensional problem of Whitehead was a great achievement of infinite-dimensional topology. More precisely, Chapman proved that a mapping between compact polyhedra $f: X \to Y$ is a simple homotopy equivalence if $f \times \mathrm{id}: X \times Q \to Y \times Q$ is homotopic to a homeomorphism [65]. The converse implication was proved earlier by West [240]. Chapman also succeeded in extending the simple homotopy theory from compact polyhedra and CW-complexes to compact ANR's [67].

In 1973 Toruńczyk proved that the direct product of a compact AR with the Hilbert space l_2 is homeomorphic to l_2 [227]. Generalizations to products of ANR's with normed vector spaces were obtained in [228]. Finally, in 1975 R.D. Edwards proved that the product of a locally compact ANR with Q is a Q-manifold (see [66]). Combining Edwards'

ANR theorem with Chapman's triangulation theorem, one immediately concludes that every compact ANR has the homotopy type of a compact polyhedron P, which answers a classical problem stated by Borsuk at the International Congress of Mathematicians held in Amsterdam in 1954.

This problem was first solved by West. He proved that every locally compact ANR X is *resolvable*, i.e. it is the image of a Q-manifold M under a cell-like mapping $f : M \to X$ [241, 242]. Since cell-like mappings between locally compact ANR's are (fine) homotopy equivalences [124], X has the homotopy type of M. If X is compact, M is also compact and, by the triangulation theorem, M has the homotopy type of a compact polyhedron. The fact that $X \times [0, 1)$ is resolvable is often referred to as Miller's theorem. Actually, R.T. Miller proved the analogous assertion for finite-dimensional ANR's and finite-dimensional manifolds [186], but the arguments were applicable to the infinite-dimensional case as well. Note the difference of behavior between Q-manifolds and n-manifolds, exemplified by the resolvability of ANR's and the lack of resolvability of homology n-manifolds (which are finite-dimensional ANR's).

In Warsaw Toruńczyk proved a remarkable characterization of Q-manifolds as locally compact ANR's having the *disjoint n-cube property*, for all n [230]. He discovered this property independently of Cannon's discovery of the disjoint disc property [47]. Actually, his result preceded Cannon's work by a few months (see p. 291 of [114]). The preprints were widely disseminated already in the beginning of 1977. However, the paper appeared only in 1980, because of the long waiting time in Fundamenta Mathematicae at that time. The strategy of Toruńczyk's proof consisted in showing that the projection $X \times Q \to X$ (under the assumptions of the theorem) fulfills Bing's shrinking criterion, which yielded a homeomorphism $X \times Q \approx X$. However, by the Edwards ANR theorem, $X \times Q$ is a Q-manifold. Alternative proofs of Toruńczyk's theorem were obtained by Edwards [96] and later by J.J. Walsh [238]. These proofs use neither the West resolution theorem nor the Edwards ANR theorem. Instead they use Miller's theorem and the scheme used in proving the characterization theorem for finite-dimensional manifolds. Toruńczyk's characterization theorem for Q-manifolds implies the Edwards ANR theorem and many other results on Q-manifolds. In 1981 Toruńczyk characterized l_2-manifolds as ANR's having the discrete-cells property [231]. An alternative proof was given in [21]. Toruńczyk also considered the characterization of nonseparable Hilbert space manifolds and solved an old problem by proving that the weight of an infinite-dimensional Fréchet space determines its topological type.

There exist elementary examples of cell-like mappings $f : X \to Y$ between metric compacta, which are not homotopy equivalences. A much deeper fact is the existence of cell-like mappings which are not shape equivalences. The first such example was described by J.L. Taylor [224], who used sophisticated algebraic topology [1, 226]. In this example X is not an ANR and $Y = Q$. There exist similar examples, where $X = Q$ and Y is not an ANR [140]. At this point it was natural to ask whether the cell-like image of a compact finite-dimensional ANR must always be an ANR? It follows from a result of George Kozlowski [149] that this is equivalent to the following question. Must a cell-like image Y of a compact finite-dimensional ANR X be finite-dimensional? This problem proved to be very difficult and for a number of years defied the efforts of many topologists.

Finally, the problem was answered negatively. First it was proved that the following two problems are equivalent: (i) Does there exist a finite-dimensional metric compactum, which admits an infinite-dimensional cell-like image? (ii) Does there exist an infinite-dimensional

Thomas A. Chapman (born in 1940 in Mt. Hope, West Virginia) obtained his Ph.D. in 1970 at Louisiana State University from Anderson. Robert Duncan Edwards (born in 1942 in Freeport, New York) obtained his Ph.D. in 1969 from the University of Michigan under James Kister. Ross Geoghegan (born in 1943 in Dublin, Ireland) obtained his Ph.D. in 1968 at Cornell University from David Wilson Henderson. James Earl West (born in 1944 in Grinnell, Iowa) obtained his Ph.D. in 1967 at Louisiana State University from Anderson. Henryk Toruńczyk (born in 1945 in Warsaw) obtained his Ph.D. in 1971 in Warsaw from Czesław Bessaga. Steven Charles Ferry (born in 1947 in Takoma Park, Maryland) obtained his Ph.D. in 1973 from Morton Brown at the University of Michigan. John Joseph Walsh (born in 1948 in Helena, Montana) obtained his Ph.D. in 1973 at the State University of New York in Binghamton from Louis McAuley.

metric compactum X with finite (integral) cohomological dimension $\dim_{\mathbb{Z}} X < \infty$? The latter was a more than 50 years old unsolved problem of P.S. Aleksandrov. The equivalence of the two questions was announced in 1978 by R.D. Edwards in an abstract in the Notices of the American Mathematical Society [97]. In 1981 J.J. Walsh published a proof in [237] with acknowledgement to Edwards. A construction described in this proof proved to be very useful in cohomological dimension theory and is usually referred to as the Edwards–Walsh complex. In 1988 Aleksandr Nikolaevich Dranishnikov in Moscow [83, 84] (a student of E.V. Shchepin born in 1958) solved the Aleksandrov problem by producing an infinite-dimensional metric compactum X having $\dim_{\mathbb{Z}} X = 3$. He used the Edwards–Walsh complex and some sophisticated computations in reduced complex K-theory with mod p coefficients [4, 45]. It was then easy to obtain a cell-like mapping $f : S^7 \to Y$ with $\dim Y = \infty$.

An important strengthening of cell-like mappings are the *hereditary shape equivalences*, i.e. proper mappings $f : X \to Y$, which have the property that, for every closed subset $B \subseteq Y$, the restriction of f to $A = f^{-1}(B)$ is a shape equivalence $f|A : A \to B$. It was proved by Kozlowski [149] that the image of a compact ANR under a hereditary shape equivalence is always an ANR. Kozlowski's influential paper was never published. According to its author, the referee (Trans. Amer. Math. Soc.) required too many changes.

Research in the theory of retracts was also going on in Moscow, especially in Smirnov's seminar. Yu.M. Smirnov, a well-known general topologist, started his seminar in 1953. In the beginning it was devoted to general and infinite-dimensional topology. Later it included the theory of retracts and shape. Yu.T. Lisitsa, a member of Smirnov's seminar, successfully applied factorization techniques to problems concerning the extension of mappings. In particular, he obtained extension theorems for mappings into LC^{∞}-spaces, which are analogues of Dugundji's theorems for mappings into LC^{n}-spaces. Moreover, he showed that ANR's for metric spaces are always ANR's for the class of M-paracompact spaces, i.e. Hausdorff spaces, which admit perfect mappings onto metric spaces [166]. S.A. Bogatyĭ, another member of the seminar, studied various types of approximate retracts, especially from the point of view of shape theory [24]. Smirnov and his group devoted a number of papers to equivariant theory of retracts [218, 219, 6, 7, 170], a topic initially studied by J.W. Jaworowski [133, 134]. E.V. Shchepin obtained the surprising result that an ANR for the class of compact Hausdorff spaces must be either infinite-dimensional or metrizable [213]. The proof uses results on uncountable inverse systems of compacta, which he developed in his Ph.D. thesis.

Yuriĭ Mikhailovich Smirnov, professor at Moscow State University, was born in Kaluga in 1921. He began his studies at Moscow State University in 1939. He first belonged to the seminar of A.N. Kolmogorov (1903–1987). He became Alexandrov's student when, on Kolmogorov's recommendation, he was assigned to Aleksandrov to help him write his papers (Aleksandrov had a very poor eyesight). Smirnov's studies were interrupted by the second world war, which he spent in the navy. Returning from the war to the University, he defended his candidate's thesis in 1951 and his D.Sc. thesis in 1958. Yuriĭ Trofimovich Lisitsa was born near Bershad' in Ukraine in 1947. He defended his candidate's thesis in 1973 at Moscow State University. Eugeniĭ Vitalevich Shchepin was born in Moscow in 1951. He was the last student of Aleksandrov. At Moscow State University he defended the candidate's thesis in 1977 and the D.Sc. thesis in 1979.

Recent advances in cohomological dimension theory led to the formation of a new area of topology, called *extension theory*. According to a classical theorem on the (covering) dimension, $\dim X \leqslant n$ if and only if every mapping $f : A \to S^n$, defined on a closed subset A of X, extends to a mapping $\tilde{f} : X \to S^n$. Similarly, for the cohomological dimension with coefficients in G, one has $\dim_G X \leqslant n$ provided every mapping $f : A \to K(G, n)$ into the Eilenberg–MacLane complex $K(G, n)$ extends to a mapping $\tilde{f} : X \to K(G, n)$. More generally, in extension theory one considers the problem of extending mappings into metric simplicial complexes and CW-complexes. This unifies and generalizes the theories of covering and cohomological dimensions [92].

2. Theory of shape

It is generally considered that shape theory was founded in 1968, when Borsuk published his well-known paper on the homotopy properties of compacta [34]. Borsuk's starting point was the observation that many theorems in homotopy theory are valid only for spaces with good local behavior, e.g., manifolds, CW-complexes, ANR's, but fail when applied to spaces like metric compacta. A simple example of this phenomenon is the already mentioned Whitehead theorem that a weak homotopy equivalence between connected CW-complexes is a homotopy equivalence.

An example showing the failure of Whitehead's theorem for metric compacta is provided by the mapping $f : X \to Y$, where X is the *Warsaw circle* and $Y = \{*\}$ is a point. The Warsaw circle, an object popular in shape theory, is the planar continuum obtained from the closure of the graph of the function $\sin(1/t)$, $t \in (0, 1/\pi]$, by identifying the points $(0, 1)$ and $(1/\pi, 0)$. The mapping f is a weak homotopy equivalence, because all the homotopy groups of the Warsaw circle vanish. Nevertheless, f is not a homotopy equivalence.

To overcome such difficulties, caused by local irregularities of spaces, Borsuk considered metric compacta embedded in the Hilbert cube Q (more generally, in a fixed absolute retract). Instead of mappings $f : X \to Y$ between such compacta, he considered *fundamental sequences* $(f_n) : X \to Y$, i.e. sequences of mappings $f_n : Q \to Q$, $n = 1, 2, \ldots$, such that, for every neighborhood V of Y in Q, there exist a neighborhood U of X in Q and an integer m such that $f_n(U) \subseteq V$, for $n \geqslant m$. Moreover, the restrictions $f_n|U$ and $f_{n'}|U$ are homotopic in V, for $n, n' \geqslant m$. Fundamental sequences compose by composing their components, i.e. $(g_n)(f_n) = (g_n f_n)$. Two fundamental sequences (f_n), (f'_n) are considered *homotopic* provided every V admits a U and an m such that $f_n|U \simeq f'_n|U$ in V, whenever $n \geqslant m$. Homotopy of fundamental sequences is an equivalence relation and the homotopy classes $[(f_n)]$ compose by composing their representatives, i.e. $[(g_n)][(f_n)] = [(g_n)(f_n)]$. In this way one obtains a category, whose objects are compacta in Q and the morphisms are homotopy classes of fundamental sequences. Since arbitrary metric compacta embed in Q, one readily extends this category to an equivalent category Sh(CM), whose objects are all metric compacta. This is *Borsuk's shape category.*

Every mapping $f : X \to Y$ induces a fundamental sequence, whose homotopy class depends only on the homotopy class of f. In this way one obtains a functor S: Ho(CM)$\to$ Sh(CM) from the homotopy category of metric compacta to Borsuk's shape category, called the *shape functor*. Compacta X, Y of the same homotopy type have the same shape, $\mathrm{sh}(X) = \mathrm{sh}(Y)$, i.e. are isomorphic objects of Sh(CM). Borsuk showed that, for a compact ANR Y, shape morphisms $F : X \to Y$ are in one-to-one correspondence with the homotopy classes of mappings $X \to Y$. Therefore, for compact ANR's, shape coincides with homotopy type. The Warsaw circle and the circle S^1 are examples of metric continua which have different homotopy types, but the same shape.

Borsuk's work on shape theory also had its precedents. These include cell-like spaces and cell-like mappings, i.e. *property* UV^∞, as well as its finite analogue, the *property* UV^n. They also include the Vietoris and the Čech homology (cohomology) groups [234, 57]. D.E. Christie's Ph.D. thesis, written in Princeton under Lefschetz's supervision, contains the beginnings of ordinary and strong shape theories [70]. In particular, Christie's homotopy groups coincide with Borsuk's shape groups. The 1-dimensional shape group was discovered even before [148]. The Brasilian topologist Elon L. Lima, a student of Edwin H. Spanier (1921–1996), generalized the Spanier–Whitehead duality to compact subsets of the sphere, by introducing a stable shape category [162]. However, in his paper no attempt was made to develop the shape category. Lima's work was "discovered" by the shape-theorists with considerable delay.

Undoubtedly, many topologists became aware of Borsuk's work on shape theory after he presented his ideas and results in Baton Rouge, Louisiana, in 1967, during a symposium on infinite dimensional topology (the proceedings were published only in 1972) and in Hercegnovi (former Yugoslavia) in 1968, during an international conference on topology. At the second of these events Borsuk used for the first time the suggestive term *shape* [35].

Shortly after Borsuk's talks and seminal papers on shape theory [34–38], an avalanche of articles on this new branch of topology appeared. By 1980 the literature on shape theory already consisted of about 400 papers. Around the world, groups of shape theorists were formed. Three specialized conferences, organized in Dubrovnik in 1976, 1981 and 1986 (Volumes 870 and 1283 of the Springer Lecture Notes in Mathematics) also contributed to the quick growth of shape theory.

In the initial period Warsaw was the center of activities in shape theory and the seat of the Borsuk group, which included J. Dydak, S. Godlewski, W. Holsztyński, A. Kadlof, J. Krasinkiewicz, Krystyna Kuperberg, P. Minc, Maria Moszyńska, S. Nowak, Hanna Patkowska, S. Spież, M. Strok, A. Trybulec.

In the US the first contributions to shape theory were made by Jack Segal, Professor at the University of Washington in Seattle (born in Philadelphia in 1934, Ph.D. in 1960 at the University of Georgia from M.K. Fort, Jr.), R.H. Fox (a well-known specialist in knot theory) and T.A. Chapman, Professor at the University of Kentucky. They were quickly joined by Billy Joe Ball (born in 1925, died in Austin, Texas in 1996) and R.B. Sher (born in Flint, Michigan in 1939) (Athens, Georgia), J.E. Keesling (born in 1942) and Philip Bacon (born in Chicago in 1929, died in Gainesville 1991) (Gainesville, Florida), R. Geoghegan and D.A. Edwards (Binghamton, New York), H.M. Hastings (Hempstead, New York), R.C. Lacher (Tallahassee, Florida), L.R. Rubin (Norman, Oklahoma), T.B. Rushing (born in Marshville, N. Carolina in 1941, died in Salt Lake City in 1998) (Salt Lake City, Utah), J.B. Quigley (Bloomington, Indiana), D.S. Coram and P. F. Duvall (Stillwater, Oklahoma), L.S. Husch (Knoxville, Tennessee), S. Ferry (Lexington, Kentucky), F.W. Cathey and G. Kozlowski (Seattle, Washington), G.A. Venema (Grand Rapids, Michigan) and many others.

In Moscow, since 1924, Aleksandrov conducted a seminar on topological spaces and dimension theory. Smirnov was a member of Aleksandrov's seminar and from 1953 to 1987 had his own seminar. Since 1970 the name of the seminar was *Seminar for shape theory and retracts.* Among the participants interested in shape theory and related areas were V.V. Agaronian, S. Antonian, S.A. Bogatyĭ, A.I. Bykov, A.Ch. Chigogidze, V.A. Kalinin, S.S. Kotanov, B.T. Levshenko, Yu.T. Lisitsa, I.S. Rubanov, A.P. Shostak, E.G. Sklyarenko, G. Skordev. Smirnov's group especially studied FANR's and related spaces as well as equivariant shape theory. In the Soviet Union contributions to shape theory were also made in Tbilisi, Georgia, by Z.R. Miminoshvili, a student of L.D. Mdzinarishvili (who in his turn was a student of G.S. Chogoshvili (1914–1998), the leading topologist in Georgia). Research in shape theory and related areas was also done in Novosibirsk by V.I. Kuz'minov, I.A. Shvedov, M.A. Batanin.

In Japan contributions to shape theory came from Kiiti Morita (1915–1995), the founder of general topology in Japan (dimension theory, product spaces) and from the group around Yukihiro Kodama at the University of Tsukuba. Kodama's group included H. Fukaishi, H. Hosokawa, H. Kato, K. Kawamura, A. Koyama, J. Ono, K. Sakai, K. Tsuda, T. Watanabe, T. Yagasaki, K. Yokoi.

The shape group in Zagreb (earlier Yugoslavia, now Croatia) was led by Sibe Mardešić (born in 1927 in Bergedorf near Hamburg, Germany). It included Z. Čerin, Q. Haxhibeqiri, K. Horvatić, I. Ivanšić, Vlasta Matijević, N. Šekutkovski, Š. Ungar, N. Uglešić. In Germany, shape theorists were led by Friedrich Wilhelm Bauer, Professor in Frankfurt a.M (born in Berlin in 1932). His group included B. Günther, P. Mrozik, H. Thiemann. In Great Britain the first contributions to shape theory were made by Timothy Porter, Professor at the University of Wales in Bangor (born in Abergavenny, Gwent in 1947). Further contributions were made by Allan Calder from Birckbeck College in London. In France shape

K. Morita was born in Hamamatsu-shi, Shizuoka. He studied at Tokyo Higher Normal School and Tokyo University of Science and Literature. He defended his Ph.D. thesis in 1950 at the University of Osaka. However, he was essentially a self-taught topologist. He was Professor at the Tokyo University of Education, which later became the University of Tsukuba. Morita also worked in algebra (module and ring theory). Y. Kodama was born in Tsuruoka in 1929. He obtained his B.Sci. from Tokyo University of Literature and Science in 1950 and his Ph.D. from Tokyo University of Education in 1960, under Morita. For his work in topology he was primarily inspired by studying the papers of Aleksandrov and Borsuk. He was Professor at Tsukuba University until his retirement in 1993.

S. Mardešić obtained his Ph.D. in 1957 from the University of Zagreb. He is essentially a self-taught topologist, influenced primarily by the work of Aleksandrov and Borsuk. F.W. Bauer obtained his Ph.D. in 1955 in Frankfurt a.M. In his work he was primarily influenced by W. Franz, P.S. Aleksandrov and S. MacLane, and considers himself a member of the Aleksandrov school. T. Porter obtained his Ph.D. from the University of Sussex in 1972. J.-M. Cordier obtained his doctorat d'état from University Paris 7 in 1987. J.M.R. Sanjurjo obtained his Ph.D. in Madrid in 1979 under the supervision of J.M. Montesinos. Being a knot-theorist, Montesinos came in touch with shape theory through Fox.

theory began with Jean-Marc Cordier and Dominique Bourn from the University of Picardie in Amiens. The Spanish shape group was led by José M.R. Sanjurjo, Professor at the Complutense University in Madrid (born in Madrid in 1951). His group included A. Giraldo, V.F. Laguna, M.A. Morón, F.R. Ruiz del Portal. Some shape theory was also done in Belgium (R.W. Kieboom), Canada (L. Demers), Italy (E. Giuli, L. Stramaccia, A. Tozzi), Mexico (Mónica Clapp, R. Jimenez, L. Montejano, Sylvia de Neymet), Romania (I. Pop), Switzerland (H. Kleisli, C. Weber).

Jack Segal spent the academic year 1969/70 in Zagreb. The result of this visit was joint work with Mardešić, generalizing Borsuk's shape theory to compact Hausdorff spaces [178, 179]. The new description of shape was based on a systematic use of inverse systems. Every compact Hausdorff space X can be represented as the inverse limit of a cofinite inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of compact polyhedra (or compact ANR's). Shape morphisms $F: X \to Y$ are given by *homotopy classes* of *homotopy mappings* $(f, f_\mu): \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, p_{\mu\mu'}, M)$. The latter consist of an increasing function $f: M \to \Lambda$ and a family of mappings $f_\mu: X_{f(\mu)} \to Y_\mu$ such that, for $\mu_0 \leqslant \mu_1$, the following diagram commutes up to homotopy

$$\begin{tikzcd} X_{f(\mu_0)} \arrow[d, "f_{\mu_0}"'] & X_{f(\mu_1)} \arrow[l, "p_{f(\mu_0)f(\mu_1)}"'] \arrow[d, "f_{\mu_1}"] \\ Y_{\mu_0} & Y_{\mu_1} \arrow[l, "q_{\mu_0\mu_1}"] \end{tikzcd} \quad . \tag{1}$$

Two homotopy mappings (f', f'_μ), (f'', f''_μ) are considered *homotopic* if there exists an increasing function $f \geqslant f', f''$ such that $f'_\mu p_{f'(\mu)f(\mu)} \simeq f''_\mu p_{f''(\mu)f(\mu)}$. Equivalence with the Borsuk approach was proved using inverse systems which consist of a decreasing sequence of compact ANR-neighborhoods of X in Q and of inclusion mappings.

While Borsuk's approach was rather geometric, the inverse system approach was more categorical and led quickly to further generalizations. In 1972 Fox generalized Borsuk's approach in a different direction, i.e. to arbitrary metric spaces X [109]. He embedded X as a closed subset in a suitable absolute retract L and used inclusion systems of ANR-neighborhoods of X in L. Both generalizations were unified in the papers by Mardešić [171] and K. Morita [191], where the general shape category Sh(Top) of arbitrary topological spaces was defined. Morita allows his systems $\boldsymbol{X}$ to be *homotopy systems*, i.e. the usual conditions on bonding mappings $p_{\lambda\lambda'}: X_{\lambda'} \to X_\lambda$ and projections $p_\lambda: X \to X_\lambda$ are replaced by homotopy conditions $p_{\lambda\lambda'}p_{\lambda'\lambda''} \simeq p_{\lambda\lambda''}$, $p_{\lambda\lambda'}p_{\lambda'} \simeq p_\lambda$, $\lambda \leqslant \lambda' \leqslant \lambda''$. Moreover, some theorems from [179] now became conditions (M1), (M2), which are part of the definition of a system being *associated* with a space:

(M1) For every mapping $f : X \to P$ to a polyhedron (or ANR) P, there exist a $\lambda \in \Lambda$ and a mapping $f_\lambda : X_\lambda \to P$ such that $f_\lambda p_\lambda \simeq f$.

(M2) For every $\lambda \in \Lambda$ and mappings $f_\lambda, f'_\lambda : X_\lambda \to P$ such that $f_\lambda p_\lambda \simeq f'_\lambda p_\lambda$, there exists an index $\lambda' \geqslant \lambda$ such that $f_\lambda p_{\lambda\lambda'} \simeq f'_\lambda p_{\lambda\lambda'}$.

Morita proved that the *Čech system,* formed by the nerves of all normal coverings of X, is a homotopy system associated with X [192]. In the terminology first used in algebraic geometry [116], shape morphisms are given by morphisms $\boldsymbol{X} \to \boldsymbol{Y}$ from the category pro-Ho(Top), where Ho(Top) denotes the homotopy category of topological spaces.

One of the first successful applications of shape theory is Fox's theory of *overlays,* a modification of covering spaces [109]. The classical theorem of covering space theory asserts that n-fold covering spaces of a connected arcwise locally connected and semilocally 1-connected space X are in a one-to-one correspondence with the classes of homomorphisms of the fundamental group $\pi_1(X)$ into the symmetric group Σ_n, where two homomorphisms ϕ, ψ belong to the same class provided there exists an inner automorphism $\theta : \Sigma_n \to \Sigma_n$ such that $\phi = \theta\psi$. Fox's shape theoretic version of the theorem, refers to overlays of arbitrary metric spaces X (embedded in some ANR). However, the fundamental group $\pi_1(X)$ has to be replaced by the *fundamental pro-group* $\pi_1(\boldsymbol{X}, *)$, the inverse system of fundamental groups of ANR-neighborhoods of X.

Further significant successes of shape theory were the shape-theoretic versions of the theorems of Whitehead, Hurewicz and Smale. The statements of these results also use pro-groups, i.e. inverse systems of groups. Application of the singular homology functor $H_m(.\,; G)$ to $\boldsymbol{X}$ yields an inverse system of Abelian groups $H_m(\boldsymbol{X}; G) = (H_m(X_\lambda; G), p_{\lambda\lambda'*}, \Lambda)$, called the m-th-*homology pro-group* of $\boldsymbol{X}$. Similarly, for systems of pointed spaces $(\boldsymbol{X}, *)$, one defines the m-th-*homotopy pro-group* $\pi_m(\boldsymbol{X}, *)$. If $\boldsymbol{X}$ and $(\boldsymbol{X}, *)$ are systems of ANR's associated with the space X and $(X, *)$, respectively, then these pro-groups do not depend on the choice of the associated systems. Moreover, they are shape invariants of X and $(X, *)$, respectively. The inverse limit $\check{H}_m(X; G) = \lim H_m(\boldsymbol{X}; G)$ is the *Čech homology group.* The *shape groups* $\check{\pi}_m(X, *) = \lim \pi_m(\boldsymbol{X}, *)$, were first defined in [70]. One should keep in mind that the Čech groups and the shape groups give less information about the space than the corresponding homology and homotopy pro-groups.

The most general version of the Whitehead theorem in shape theory is due to K. Morita [190]. It asserts that a morphism of pointed shape $F : (X, *) \to (Y, *)$ between finite-dimensional topological spaces is a *shape equivalence,* i.e. an isomorphism of *pointed shape* if and only if it induces isomorphisms of all homotopy pro-groups $F_\# : \pi_m(\boldsymbol{X}, *) \to \pi_m(\boldsymbol{Y}, *)$. In contrast to the classical Whitehead theorem, there are no restrictions on the local behavior of the spaces involved. Morita's result was preceded by less general versions of the theorem, obtained by Moszyńska [193] and Mardešić [172]. The restriction to finite dimensions cannot be omitted. A counterexample was obtained in [82], using a metric continuum defined by D.S. Kahn [135]. For every odd prime p, one considers the CW-complex X_0, obtained by attaching a $(2p+1)$-cell to S^{2p} by a mapping of degree p. One defines X_{n+1} as the $(2p-2)$-fold suspension $\Sigma^{2p-2}(X_n)$, $n \geqslant 0$. A particular mapping $f_1 : X_1 \to X_0$ is chosen. For $n > 1$, one defines mappings $f_n : X_n \to X_{n-1}$ by putting $f_n = \Sigma^{2p-2}(f_{n-1})$. The Kahn continuum is the limit of the inverse sequence defined

Stanisław Spież (born in Kalisz, Poland in 1944), Sławomir Nowak (born in Sosnowiec, Poland in 1946) and Jerzy Dydak (born in Brzozów, Poland in 1951) obtained their Ph.D. degrees from the University of Warsaw in 1973, 1973 and 1975, respectively. They were Borsuk's students. Dydak moved to US in 1982.

by the spaces X_n and by the mappings f_n. The crucial property that all the compositions $f_i \circ \cdots \circ f_j$, $i < j$, are essential mappings depends on deep results in homotopy theory [1, 226].

In the Whitehead theorem mentioned above the restriction to finite-dimensional spaces can be replaced by the weaker restriction to spaces of finite *shape dimension* sd (also called *fundamental dimension* and denoted by Fd). This is a numerical shape invariant introduced by Borsuk [36]. An extensive study of this notion was carried out by Polish topologists S. Nowak [196] and S. Spież [220, 221].

The shape-theoretic Hurewicz theorem involves homology pro-groups. One assumes that X is a $(n-1)$-*shape connected* space, $n \geqslant 2$, i.e. its homotopy pro-groups $\pi_m(\boldsymbol{X}, *)$ vanish, for $m \leqslant n-1$. One concludes that the corresponding homology pro-groups $H_m(\boldsymbol{X}; \mathbb{Z})$ vanish and there exists a natural isomorphism $\phi_n : \pi_n(\boldsymbol{X}, *) \to H_n(\boldsymbol{X}; \mathbb{Z})$ of the n-th-pro-groups. The general result is due to Morita [190]. Earlier versions involving shape groups were obtained by M. Artin and B. Mazur [10] and K. Kuperberg [152]. A Hurewicz theorem involving Steenrod homology is due to Y. Kodama and A. Koyama [146] and to Yu.T. Lisitsa [168].

The classical Smale theorem is the homotopy version of a theorem of Vietoris concerning cell-like mappings of compacta [217]. The shape-theoretic Smale theorem was proved by J. Dydak [89, 91] and asserts that, for metric compacta, every cell-like mapping induces isomorphisms of homotopy pro-groups $f_{\#} : \pi_n(\mathbf{X}, *) \to \pi_n(\boldsymbol{Y}, *)$, for all n and all base-points. Consequently, if sd X, sd $Y < \infty$, the Whitehead theorem applies and f is a shape equivalence.

Among the most important contributions of Borsuk to shape theory is the introduction of two shape invariant classes of metric compacta, the *fundamental absolute neighborhood retracts* FANR's [36] and *movable compacta* [37]. X is an FANR provided, for any compact metric space Y containing X, there exist a closed neighborhood U of X in Y and a *shape retraction* $R : U \to X$, i.e. a shape morphism which is a left shape inverse of the inclusion mapping $i : X \to U$, $RS[i] = \mathrm{id}_X$. Clearly, every compact ANR is an FANR. Many results from the theory of retracts have their analogues in the theory of shape. For example, if X is *shape dominated* by X' (i.e. there exist shape morphisms $f : X \to X'$ and $g : X' \to X$ such that $gf = \mathrm{id}_X$) and X' is an FANR, then X is also an FANR. This implies that FANR's coincide with metric compacta which are shape dominated by compact polyhedra.

A compact space X, embedded in the Hilbert cube Q, is movable provided every neighborhood U of X in Q admits a neighborhood U' of X such that, for any neighborhood $U'' \subseteq U$ of X, there exists a homotopy $H : U' \times I \to U$ with $H(x, 0) = x$, $H(x, 1) \in U''$, for all $x \in U'$. In other words, sufficiently small neighborhoods of X can be deformed arbitrarily close to X. Borsuk proved that this remarkable property is a shape invariant. In a subsequent paper, he characterized FANR's by a similar property, called *strong movability* [38]. From its definition it is clear that FANR's are always movable. In fact, Borsuk introduced movability as a tool needed to detect that some compacta, e.g., the solenoids, are not FANR's. Borsuk also introduced the notion of *n-movability* and proved that LC^{n-1} compacta are always n-movable [39]. A compactum $X \subseteq Q$ is n-movable provided every neighborhood U of X in Q admits a neighborhood U' of X in Q such that, for any neighborhood $U'' \subseteq U$ of X, any compactum K of dimension $\dim K \leqslant n$ and any mapping $f : K \to U'$, there exists a mapping $g : K \to U''$, such that f and g are homotopic in U. Clearly, if a compactum X is n-movable and $\dim X \leqslant n$, then X is movable. The notion of n-movability was the beginning of *n-shape theory*, which was especially developed in the papers of A.Ch. Chigogidze [69]. The n-shape theory is an important tool in the theory of n-dimensional Menger manifolds, developed by M. Bestvina [19].

Further studies revealed the importance of *pointed* FANR's and *pointed movability*. For example, the union of two pointed FANR's, whose intersection is a pointed FANR, is again a pointed FANR [94]. The main protagonists of this research were D.A. Edwards, R. Geoghegan, H.M. Hastings, A. Heller and J. Dydak. It was shown in [99, 101] that connected pointed FANR's coincide with *stable continua*, i.e. continua having the shape of a polyhedron. In general one cannot achieve that this polyhedron be compact. This is because there exist noncompact polyhedra P, which are homotopy dominated by compact polyhedra, but do not have the homotopy type of a compact polyhedron [236]. Edwards and Geoghegan [100] defined a Wall obstruction $\sigma(X)$ for FANR's X and they showed that X has the shape of a compact polyhedron if and only if $\sigma(X) = 0$. Since $\sigma(X)$ is an element of the reduced projective class group $\tilde{K}^0(\check{\pi}_1(X, *))$ of the first shape group $\check{\pi}_1(X, *)$, this result linked shape theory to K-theory.

The question whether every FANR is a pointed FANR eluded the efforts of shape theorists for several years. Finally, in 1982, Hastings and Heller proved that this is always the case [122]. The crucial step in their proof is a purely homotopy theoretic result. This is the theorem that on a finite-dimensional polyhedron X every homotopy idempotent $f : X \to X$ splits, i.e. $f^2 \simeq f$ implies the existence of a space Y and of maps $u : Y \to X$, $v : X \to Y$, such that $vu \simeq 1_Y$, $uv \simeq f$. The proof uses nontrivial combinatorial group theory as well as the spectral sequence of a covering mapping. More precisely, it uses a particular group G and a particular homomorphism $\phi : G \to G$, which induces an unsplittable homotopy idempotent $f : K(G, 1) \to K(G, 1)$ of Eilenberg–Mac Lane complexes. It also uses the fact that the construction is universal in the sense that whenever $f' : X \to X$ is an unsplit homotopy idempotent, then there is an injection $G \to \pi_1(X)$, which is equivariant with respect to $f_\#$ and $f'_\#$. The group G itself has been considered before by R.J. Thompson (unpublished). Parts of the argument were discovered independently by P. Minc, by J. Dydak [90] and by P. Freyd and A. Heller (unpublished). The question whether movable continua are always pointed movable is still open.

For movable spaces various shape-theoretic results assume simpler form. For example, if $f : (X, *) \to (Y, *)$ is a pointed shape morphism between pointed movable metric continua, which induces isomorphisms of shape groups $f_\# : \check{\pi}_k(X, *) \to \check{\pi}_k(Y, *)$, for all k and if the spaces X, Y are finite-dimensional, then f is a pointed shape equivalence. This is a consequence of the shape-theoretic Whitehead theorem and the fact that such an f induces isomorphisms of homotopy pro-groups $\boldsymbol{\pi}_k(\boldsymbol{X}, *) \to \boldsymbol{\pi}_k(\boldsymbol{Y}, *)$ [141, 91].

In 1972 a new direction in shape theory was inaugurated by Chapman. He applied methods of infinite-dimensional topology to the study of shape of metric compacta [62]. More precisely, he considered compacta X which are *Z-embedded* in the Hilbert cube Q, i.e. have the property that there exist mappings $f : Q \to Q$, which are arbitrarily close to the identity but their image $f(Q)$ misses X. This condition, introduced by R.D. Anderson [5], implies tameness and unknottedness of compacta and proved to be fundamental in the development of the theory of Q-manifolds [66]. Chapman's *complement theorem* asserts that two compacta X, Y, embedded in Q as Z-sets, have the same shape if and only if their complements $Q\backslash X$, $Q\backslash Y$ are homeomorphic. Chapman also exhibited an isomorphism of categories $T:\mathcal{WP} \to \mathcal{S}$. The domain of T is the *weak proper homotopy category* of complements $M = Q\backslash X$ of Z-sets X of Q. Morphisms of $\mathcal{WP}$ are equivalence classes of proper mappings $f:M \to N = Q\backslash Y$. Two such mappings $f, g:M \to N$ are considered equivalent provided every compact set $B \subseteq N$ admits a compact set $A \subseteq M$ and a homotopy $H:M \times I \to N$ such that H connects f to g and $H((M\backslash A) \times I) \subseteq N\backslash B$. The codomain of T is the restriction of the shape category Sh(CM) to Z-sets X of Q. On objects $M = Q\backslash X$ of $\mathcal{WP}$ one has $T(M) = Q\backslash M = X$.

Subsequently, Chapman published a second paper, which contained a finite-dimensional complement theorem, i.e. a theorem where the ambient space was the Euclidean space [63]. This paper had a strong geometric flavor and immediately attracted the attention of a number of specialists in geometric topology, in particular in PL-topology, who produced a series of finite-dimensional complement theorems. In most of these theorems one assumes that X and Y are "nicely" embedded in the Euclidean space $\mathbb{R}^n$ and satisfy the appropriate dimensional conditions. The conclusion is that X and Y have the same shape if and only if their complements $R^n\backslash X$, $R^n\backslash Y$ are homeomorphic. The most general of the results obtained is the complement theorem from [132]. It assumes that X and Y are shape r-connected, $\operatorname{sd} X = \operatorname{sd} Y = k$, $n - k \geqslant 4$ and $n \geqslant \max\{5, 2k + 2 - r\}$. The "niceness" condition is the *inessential loops condition* ILC, introduced by G.A. Venema [233]. A compactum $X \subseteq \mathbb{R}^n$ satisfies ILC provided every open neighborhood U of X in $\mathbb{R}^n$ admits an open neighborhood V of X in U, such that each loop in $V\backslash X$, which is null-homotopic in V, is also null-homotopic in $U\backslash X$. This condition was preceded by McMillan's *cellularity criterion* CC [183]. Complement theorems in more general ambient spaces and different categories were studied extensively by P. Mrozik [195]

A compact metric space X *embeds up to shape* in a space Y provided Y contains a metric compactum X' such that $\operatorname{sh}(X) = \operatorname{sh}(X')$. L.S. Husch and I. Ivanšić obtained several interesting results concerning this notion. In particular, they showed that every r-shape connected and pointed $(r+1)$-movable compactum X with $\operatorname{sd}(X) = k$, $k \geqslant 3$, embeds up to shape in $\mathbb{R}^{2k-r}$ [131].

Based on Quillen's homotopical algebra [202], Edwards and Hastings introduced a homotopy category of inverse systems, denoted by Ho(pro-Top). It is obtained from the category pro-Top by localization at level homotopy equivalences. Using this category instead of pro-Ho(Top), they defined a strong shape category SSh(CM) of compact metric spaces. Strong shape has distinct advantages over shape, e.g., Edwards and Hastings showed that the analogue of Chapman's category isomorphism theorem assumes a more natural form. It asserts the existence of an isomorphism $T:\mathcal{P} \to \mathcal{SS}$, between *proper homotopy category* $\mathcal{P}$ of complements $M = Q\backslash X$ of Z-sets X of Q and the restriction $\mathcal{SS}$ of the strong shape category SSh(CM) to Z-sets of Q [102, 147]. The strong shape category for metric compacta was first defined by J.B. Quigley, a student of J. Jaworowski at the University of Indiana in Bloomington [201].

Through efforts of various authors over several years, in particular, Porter [199], Bauer [15, 16], Calder and Hastings [46], Miminoshvili [189], Cathey and Segal [48], Lisitsa [167], Lisica and Mardešić [163, 164], Dydak and Nowak [93], Günther [117], David A. Edwards was born in 1946 in New York. He obtained his Ph.D. in 1971 from Columbia University in New York. Harold M. Hastings was born in 1946 in Dayton, Ohio. He obtained his Ph.D. in 1972 from Princeton University.

a *strong shape category* for topological spaces SSh(Top) was defined and so was a strong shape functor $\overline{S}:\mathrm{Ho(Top)} \to \mathrm{SSh(Top)}$. It is related to the shape functor S by a factorization $S = \overline{E}\,\overline{S}$, where $\overline{E}:\mathrm{SSh(Top)} \to \mathrm{Sh(Top)}$ is a functor which forgets part of the richer structure of strong shape.

In defining the strong shape category for arbitrary spaces, one needed a method of associating with any given space X a system of polyhedra (or ANR's) in the category Top. One way of doing this is provided by the Vietoris system [199, 118]. Another approach, used by Bauer, rigidifies a construction from [171] and associates with X a 2-category P_X. Its objects are mappings into polyhedra $g: X \to P$ and its 1-morphisms $g_1 \to g_2$ are given by a mapping $r: P_1 \to P_2$ and a homotopy ω, which connects rg_1 with g_2. The 2-morphisms are defined by homotopies of order 2. This approach was generalized to homotopies of arbitrarily high order (expressed in simplicial terms) by Günther [117].

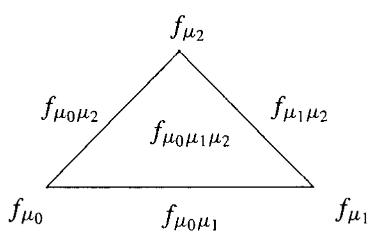
Another method is based on the notion of *resolution* of a space X [173] (more generally, on *strong expansions* [93, 117, 174]). A resolution $\boldsymbol{p}: X \to \boldsymbol{X}$ is a morphism of pro-Top which satisfies a stronger version of Morita's conditions.

(R1) Given a polyhedron P and an open covering $\mathcal{V}$ of P, any mapping $f: X \to P$ admits a $\lambda \in \Lambda$ and a mapping $h: X_\lambda \to P$ such that the mappings hp_λ and f are $\mathcal{V}$-near.

(R2) There exists an open covering $\mathcal{V}'$ of P, such that whenever, for a $\lambda \in \Lambda$ and two mappings $h, h': X_\lambda \to P$, the mappings $hp_\lambda, h'p_\lambda$ are $\mathcal{V}'$-near, then there exists a $\lambda' \geqslant \lambda$ such that the mappings $hp_{\lambda\lambda'}, h'p_{\lambda\lambda'}$ are $\mathcal{V}$-near.

To define a strong shape morphism $F: X \to Y$, it suffices to choose (cofinite) polyhedral resolutions $\boldsymbol{p}: X \to \boldsymbol{Y}$, $\boldsymbol{q}: Y \to \boldsymbol{Y}$ and a morphism $\boldsymbol{X} \to \boldsymbol{Y}$ of Ho(pro-Top).

It is an important fact that the category Ho(pro-Top) is equivalent to the coherent homotopy category CH(Top), which can be viewed as a concrete realization of the former category [163, 164]. Its morphisms are coherent homotopy classes of coherent mappings $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$. The latter consist of an increasing function $f: M \to \Lambda$ and of mappings $f_{\mu_0}: X_{f(\mu_0)} \to Y_{\mu_0}$, which make diagram (1) commutative up to a homotopy $f_{\mu_0\mu_1}: X_{f(\mu_1)} \times I \to Y_{\mu_0}$, which is also part of the structure of $\boldsymbol{f}$. For three indices $\mu_0 \leqslant \mu_1 \leqslant \mu_2$, one has homotopies $f_{\mu_0\mu_1\mu_2}: X_{f(\mu_2)} \times \Delta^2 \to Y_{\mu_0}$, where Δ^2 is the standard 2-simplex. One requires that, on the faces of Δ^2, $f_{\mu_0\mu_1\mu_2}$ is given by the mappings $f_{\mu_1\mu_2}, f_{\mu_0\mu_2}, f_{\mu_0\mu_1}$ as indicated on the following figure.



Analogous requirements are imposed on higher homotopies $f_{\mu_0 \ldots \mu_n} : X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, for all increasing sequences $\mu_0 \leqslant \cdots \leqslant \mu_n$ and all n. There are other, more sophisticated descriptions of coherent categories, due to J.M. Boardman and R.M. Vogt [23, 235], Cordier and Porter [72, 73], N. Šekutkovski [211], Batanin [14], but they all yield categories equivalent to CH(Top).

An important circle of ideas, related to strong shape, refers to *strong* or *Steenrod homology*. It was originally defined only for metric compacta [222]. Over the years, especially in former USSR, much work was done on strong homology of general spaces [215, 216]. The relation of strong homology to singular and Čech homology is similar to the relation of strong shape to homotopy and ordinary shape. For pairs of spaces (X, A), where A is normally embedded in X (e.g., if A is closed and X is paracompact), all the Eilenberg–Steenrod axioms are fulfilled. From the point of view of shape theory, the most important property of strong homology is its invariance with respect to strong shape [163, 165]. In contrast to Čech homology, Čech cohomology has a long record of successful applications. The explication lies in the fact that direct limit is an exact functor, while inverse limit is not, i.e. in general, the derived functors $\lim^n$ of lim are nontrivial. The higher limits $\lim^n H_m(X; \mathbb{Z})$ of the homology pro-groups play an important role in strong homology of spaces. Actually, there exist paracompact spaces X with $\lim^n H_m(X; \mathbb{Z}) \neq 0$, for n arbitrarily high [175]. However, if X is compact, $\lim^n H_m(X; \mathbb{Z}) = 0$, for $n \geqslant 2$ [156, 176].

Using a suitable *approximate homotopy lifting property*, D.S. Coram and P.F. Duvall have introduced *approximate fibrations* as mappings $f : X \to Y$ between ANR's, which generalize cell-like mappings and share many homotopy-theoretic properties with fibrations [71]. This class of mappings proved very useful in the study of mappings between manifolds. For mappings between metric compacta, approximate fibrations had to be replaced by *shape fibrations* [177, 250]. The definition of a shape fibration between arbitrary spaces required the notion of resolution of a mapping [173]. A very useful generalization of the latter notion was introduced by T. Watanabe, who introduced *approximate resolutions* of mappings [239]. Subsequently, a more general theory was developed in [181].

Appropriate variations of the basic ideas of shape led to new types of shape theories. In particular, there is *fibered shape* [138, 249], *equivariant shape* [8, 61], *stable shape* [197, 18], *proper shape* [12, 11], *uniform shape* [209, 188].

Generally, one expects to find applications of shape theory in problems concerning global properties of spaces having irregular local behavior. Such spaces naturally appear in many areas of mathematics. A typical example is provided by the fibers of a mapping as in the case of cell-like mappings. Other examples are given by remainders of compactifications, by sets of fixed points, by attractors of dynamical systems and by spectra of operators. In the latter, strong extraordinary homology plays and important role [137, 136, 17, 76].

Keesling has devoted a series of papers to the study of the remainder $\beta X \backslash X$ of a locally compact space X in its Čech–Stone compactification [142]. In this research he used his earlier results concerning the Čech cohomology groups of movable spaces. Recently, shape theory found applications also in the field of geometric group theory. More precisely, the boundary ∂G of a (discrete) group G is defined as a Z – set of a finite-dimensional compact AR $\tilde{X}$, such that the following two axioms hold: (i) $X = \tilde{X} \backslash Z$ admits a covering space action of G with compact quotient; (ii) The collection of translates of a compact set in X forms a null-sequence in $\tilde{X}$, i.e. for every open covering $\mathcal{U}$ of $\tilde{X}$ all but finitely many translates are $\mathcal{U}$-small. The boundary ∂G is determined up to shape, i.e. if Z_1 and Z_2 satisfy the above axioms, then $\mathrm{sh}(Z_1) = \mathrm{sh}(Z_2)$ [20].

Shape theory also led to new developments in the fixed point theorems. For every compact ANR X and every mapping $f : X \to X$, the Lefschetz number $\Lambda(f)$ is a well-defined integer. If $\Lambda(f) \neq 0$, then f has a fixed point. This well-known theorem is not true for arbitrary metric compacta, because for acyclic continua $\Lambda(f) = 1$ and they need not have the fixed point property. Nevertheless, Borsuk proved that, for an arbitrary metric compactum X, $\Lambda(f) \neq 0$ implies the existence of fixed points, provided f belongs to a certain class of mappings, called *nearly extendible mappings* [40]. Another new result asserts that the space 2^X of nonempty compacta and the space $C(X)$ of nonempty continua in a locally connected Hausdorff continuum X have the fixed point property [210]. This was known before only for Peano continua X.

As an example of application of shape theory in dynamical systems we state the following result. A finite-dimensional metric compactum embeds in a (differentiable) manifold M as an attractor of a (smooth) dynamical system on M if and only if it has the shape of a compact polyhedron [119, 208]. Another application of shape concerns the definition of the Conley index for continuous and discrete dynamical systems [205].

Shape theory has also applications in the theory of continua. For example, joinable continua were characterized as pointed 1-movable continua [150]. H. Kato successfully applied shape theory to the study of Whitney mappings of hyperspaces 2^X and $C(X)$ [139].

There are many situations, where shape itself does not apply, but its methods do. Typical examples are properties at infinity of locally compact spaces (see [58]) and proper homotopy (see [185, 200, 59]). Ideas of shape theory had a bearing on homology of groups [115]. The abstract aspects of shape led to *categorical shape theory* [75] and opened further possibilities of application, e.g., in pattern recognition [198, 74].

The approach to shape by inverse systems of polyhedra or ANR's is not the only one. A different approach, recently inaugurated by Sanjurjo [207] and further developed by Čerin [60] is based on the idea of replacing mappings by multivalued mappings, which map points into sufficiently small sets.

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