Quantum electrodynamics (QED)

based on S-58

Quantum electrodynamics is a theory of photons interacting with the electrons and positrons of a Dirac field:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma^\mu\Psi A_\mu$$

$e = -0.302822$

$\alpha = e^2/4\pi = 1/137.036$

Noether current of the lagrangian for a free Dirac field $j^\mu(x) = e\overline{\Psi}(x)\gamma^\mu\Psi(x)$

$$\partial_\mu j^\mu(x) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)$$

we want the current to be conserved and so we need to enlarge the gauge transformation also to the Dirac field:

global symmetry is promoted into local

$\Psi \to e^{-i\alpha}\Psi$

$\overline{\Psi} \to e^{+i\alpha}\overline{\Psi}$

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Gamma(x)\ ,$$

$$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x)\ ,$$

$$\overline{\Psi}(x) \to \exp[+ie\Gamma(x)]\overline{\Psi}(x)\ .$$

symmetry of the lagrangian and so the current is conserved no matter if equations of motion are satisfied

We can write the QED lagrangian as:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}D\!\!\!\!/\Psi - m\overline{\Psi}\Psi$$

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

covariant derivative

(the covariant derivative of a field transforms as the field itself)

$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x)$

$$D_\mu\Psi(x) \to \exp[-ie\Gamma(x)]D_\mu\Psi(x)$$

and so the lagrangian is manifestly gauge invariant!

Proof:

$$\begin{aligned}
D_\mu\Psi \to{}& \big(\partial_\mu - ie[A_\mu - \partial_\mu\Gamma]\big)\big(\exp[-ie\Gamma]\Psi\big) \\
={}& \exp[-ie\Gamma]\big(\partial_\mu\Psi - ie(\partial_\mu\Gamma)\Psi - ie[A_\mu - \partial_\mu\Gamma]\Psi\big) \\
={}& \exp[-ie\Gamma]\big(\partial_\mu - ieA_\mu\big)\Psi \\
={}& \exp[-ie\Gamma]D_\mu\Psi\ .
\end{aligned}$$

$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x)$

$D_\mu\Psi(x) \to \exp[-ie\Gamma(x)]D_\mu\Psi(x)$

We can also define the transformation rule for D:

$$D_\mu \to e^{-ie\Gamma}D_\mu\, e^{+ie\Gamma}$$

then

$$\begin{aligned}
D_\mu\Psi \to{}& \big(e^{-ie\Gamma}D_\mu e^{+ie\Gamma}\big)\big(e^{-ie\Gamma}\Psi\big) \\
={}& e^{-ie\Gamma}D_\mu\Psi\ ,
\end{aligned}$$

as required.

Now we can express the field strength in terms of D's:

$D_\mu \equiv \partial_\mu - ieA_\mu$

$$[D^\mu, D^\nu]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x)$$

$$F^{\mu\nu} = \tfrac{i}{e}[D^\mu, D^\nu]$$

$F^{\mu\nu} = \tfrac{i}{e}[D^\mu, D^\nu]$

$D_\mu \to e^{-ie\Gamma}D_\mu\, e^{+ie\Gamma}$

Then we simply see:

$$\begin{aligned}
F^{\mu\nu} \to{}& \tfrac{i}{e}\Big[e^{-ie\Gamma}D^\mu\, e^{+ie\Gamma}, e^{-ie\Gamma}D^\nu\, e^{+ie\Gamma}\Big] \\
={}& e^{-ie\Gamma}\Big(\tfrac{i}{e}[D^\mu, D^\nu]\Big)e^{+ie\Gamma} \\
={}& e^{-ie\Gamma}F^{\mu\nu}e^{+ie\Gamma} \\
={}& F^{\mu\nu}\ .
\end{aligned}$$

no derivatives act on exponentials

the field strength is gauge invariant as we already knew

Nonabelian symmetries

based on S-24

Let's generalize the theory of two real scalar fields:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_1\partial_\mu\varphi_1 - \tfrac{1}{2}\partial^\mu\varphi_2\partial_\mu\varphi_2 - \tfrac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - \tfrac{1}{16}\lambda(\varphi_1^2+\varphi_2^2)^2$$

to the case of N real scalar fields:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}m^2\varphi_i\varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2$$

the lagrangian is clearly invariant under the SO(N) transformation:

orthogonal matrix with det = 1

$$\varphi_i(x) \to R_{ij}\varphi_j(x) \qquad R^T = R^{-1} \qquad \det R = +1$$

lagrangian has also the Z_2 symmetry, $\varphi_i(x) \to -\varphi_i(x)$, that enlarges SO(N) to O(N)

infinitesimal SO(N) transformation:

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

antisymmetric

real

$$R^T = R^{-1} \qquad R^T_{ij} = \delta_{ij} + \underline{\theta_{ji}} \qquad R^{-1}_{ij} = \delta_{ij} \underline{- \theta_{ij}}$$

$$Im(R^{-1}R)_{ij} = Im\sum_k R_{ki}R_{kj} = 0$$

(N^2 linear combinations of Im parts = 0)

there are $\frac{1}{2}N(N-1)$ linearly independent real antisymmetric matrices, and we can write:

$$\theta_{jk} = -i\theta^a(T^a)_{jk}$$

hermitian, antisymmetric, NxN generator matrices of SO(N)

or $R = e^{-i\theta^a T^a}$.

The commutator of two generators is a lin. comb. of generators:

$$[T^a, T^b] = if^{abc}T^c$$

we choose normalization: $\mathrm{Tr}(T^aT^b) = 2\delta^{ab} \to f^{abd} = -\frac{1}{2}i\,\mathrm{Tr}\big([T^a,T^b]T^d\big)$

structure constants of the SO(N) group

e.g. SO(3):

$$(T^a)_{ij} = -i\varepsilon^{aij}$$

$$[T^a, T^b] = i\varepsilon^{abc}T^c \qquad \varepsilon^{123} = +1$$

Levi-Civita symbol

consider now a theory of N complex scalar fields:

$$\mathcal{L} = -\partial^\mu\varphi_i^\dagger\partial_\mu\varphi_i - m^2\varphi_i^\dagger\varphi_i - \tfrac{1}{4}\lambda(\varphi_i^\dagger\varphi_i)^2$$

the lagrangian is clearly invariant under the U(N) transformation:

group of unitary NxN matrices

$$\varphi_i(x) \to U_{ij}\varphi_j(x)$$

$$U^\dagger = U^{-1}$$

we can always write $U_{ij} = e^{-i\theta}\widetilde{U}_{ij}$ so that $\det\widetilde{U} = +1$.

SU(N) - group of special unitary NxN matrices

U(N) = U(1) x SU(N)

actually, the lagrangian has larger symmetry, SO(2N):

$$\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}$$

$$\varphi_j^\dagger\varphi_j = \tfrac{1}{2}(\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)$$

REVIEW

infinitesimal SU(N) transformation:

hermitian $U^\dagger = U^{-1}$

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + O(\theta^2)$$

traceless

$\det \tilde{U} = +1$

or $\tilde{U} = e^{-i\theta^a T^a}$.

$\ln \det A = \mathrm{Tr} \ln A$

there are $N^2 - 1$ linearly independent traceless hermitian matrices:

$$[T^a, T^b] = i f^{abc} T^c$$

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

e.g. SU(2) - 3 Pauli matrices

SU(3) - 8 Gell-Mann matrices

the structure coefficients are $f^{abc} = 2\varepsilon^{abc}$, the same as for SO(3)

Nonabelian gauge theory

based on S-69

Consider a theory of N scalar or spinor fields that is invariant under:

$$\phi_i(x) \to U_{ij}\phi_j(x)$$

for SU(N): a special unitary NxN matrix

for SO(N): a special orthogonal NxN matrix

In the case of U(1) we could promote the symmetry to local symmetry but we had to include a gauge field $A_\mu(x)$ and promote ordinary derivative to covariant derivative:

$$\phi(x) \to U(x)\phi(x) \qquad U(x) = \exp[-ie\Gamma(x)] \qquad D_\mu = \partial_\mu - ieA_\mu$$

$$D_\mu \to U(x) D_\mu U^\dagger(x)$$

then the kinetic terms and mass terms: $-(D_\mu\varphi)^\dagger D^\mu\varphi$, $m^2\varphi^\dagger\varphi$, $i\overline{\Psi}\not{D}\Psi$ and $m\overline{\Psi}\Psi$, are gauge invariant. The transformation of covariant derivative in general implies that the gauge field transforms as:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{e}U(x)\partial_\mu U^\dagger(x)$$

for U(1): $A_\mu(x) \to A_\mu(x) - \partial_\mu\Gamma(x)$

Now we can easily generalize this construction for SU(N) or SO(N):

an infinitesimal SU(N) transformation:

from 1 to N^2-1

from 1 to N

$$U_{jk}(x) = \delta_{jk} - ig\theta^a(x)(T^a)_{jk} + O(\theta^2)$$

gauge coupling constant

generator matrices (hermitian and traceless):

$$[T^a, T^b] = i f^{abc} T^c$$

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}$$

structure constants (completely antisymmetric)

the SU(N) gauge field is a traceless hermitian NxN matrix transforming as:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

$$U(x) = \exp[-ig\Gamma^a(x)T^a]$$

the covariant derivative is:

$$D_\mu = \partial_\mu - igA_\mu(x)$$

NxN identity matrix

or acting on a field:

$$(D_\mu\phi)_j(x) = \partial_\mu\phi_j(x) - igA_\mu(x)_{jk}\phi_k(x)$$

using covariant derivative we get a gauge invariant lagrangian

We define the field strength (kinetic term for the gauge field) as:

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu]$$

a new term

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

it transforms as:

$D_\mu \to U(x)D_\mu U^\dagger(x)$

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^\dagger(x)$$

not gauge invariant separately

and so the gauge invariant kinetic term can be written as:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$$

we can expand the gauge field in terms of the generator matrices:

$$A_\mu(x) = A^a_\mu(x) T^a$$

that can be inverted:

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}$$

$$A^a_\mu(x) = 2\,\mathrm{Tr}\, A_\mu(x) T^a$$

similarly:

$$F_{\mu\nu}(x) = F^a_{\mu\nu} T^a\,,$$

$$F^a_{\mu\nu}(x) = 2\,\mathrm{Tr}\, F_{\mu\nu} T^a\,.$$

$$F_{\mu\nu}(x) \equiv \tfrac{i}{g}[D_\mu, D_\nu]$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \longrightarrow F^c_{\mu\nu} T^c = (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) T^c - ig A^a_\mu A^b_\nu [T^a, T^b]$$

$$[T^a, T^b] = i f^{abc} T^c \longrightarrow \quad = (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu) T^c\,.$$

thus we have:

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$$

$$\mathcal{L}_{\rm kin} = -\tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu})$$

$$F_{\mu\nu}(x) = F^a_{\mu\nu} T^a$$

the kinetic term can be also written as:

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}$$

$$\mathcal{L}_{\rm kin} = -\tfrac{1}{4} F^{c\mu\nu} F^c_{\mu\nu}$$

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$$

Example, quantum chromodynamics - QCD:

$$\mathcal{L} = i\overline{\Psi}_{iI} \not{D}_{ij} \Psi_{jI} - m_I \overline{\Psi}_I \Psi_I - \tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu})$$

flavor index: up, down, strange, charm, top, bottom

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig A^a_\mu T^a_{ij}$$

1, ..., 8 gluons (massles spin 1 particles)

color index: 1, 2, 3

in general, scalar and spinor fields can be in different representations of the group, $T^a_{\rm R}$; gauge invariance requires that the gauge fields transform independently of the representation.

Group representations

based on S-70

A representation of a group is specified by a set of $D(\mathrm{R}) \times D(\mathrm{R})$ hermitian matrices that obey:

$$[T^a_{\rm R}, T^b_{\rm R}] = i f^{abc} T^c_{\rm R}$$

the dimension of the representation

structure constants (real numbers)

(the original set of NxN dimensional matrices for SU(N) or SO(N) corresponds to the fundamental representation)

taking the complex conjugate we see that $-(T^a_{\rm R})^*$ is also a representation!

- R is real if $-(T^a_{\rm R})^* = T^a_{\rm R}$ or if there is a unitary transformation $T^a_{\rm R} \to U^{-1} T^a_{\rm R} U$ that makes $-(T^a_{\rm R})^* = T^a_{\rm R}$

  e.g. fundamental reps. of SO(N)

- R is pseudoreal if it is not real but there is a transformation such that $-(T^a_{\rm R})^* = V^{-1} T^a_{\rm R} V$

  e.g. the fundamental rep of SU(2) : $-(\tfrac{1}{2}\sigma^a)^* \neq \tfrac{1}{2}\sigma^a$
  $-(\tfrac{1}{2}\sigma^a)^* = V^{-1}(\tfrac{1}{2}\sigma^a) V\,, \quad V = \sigma_2$

- if R is not real or pseudoreal then it is complex

  e.g. fundamental reps. of SU(N), N>2

  complex conjugate representation $\overline{\mathrm{R}}$ is specified by: $T^a_{\overline{\rm R}} = -(T^a_{\rm R})^*$

The adjoint representation A:

$$(T^a_{\rm A})^{bc} = -i f^{abc}$$

A is a real representation
$-(T^a_{\rm A})^* = T^a_{\rm A}$

the dimension of the adjoint representation, $D(\mathrm{A})$ = # of generators
= the dimension of the group

to see that $T^a_{\rm A}$ s satisfy commutation relations we use the Jacobi identity:

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0$$

follows from:

$$\mathrm{Tr}\, T^e\Big([[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b]\Big) = 0$$

$$[T^a, T^b] = i f^{abc} T^c$$

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}$$

$$(-i f^{abd})(-i f^{cde}) - (-i f^{cbd})(-i f^{ade}) = i f^{acd}(-i f^{dbe})$$

$$(T^a_{\rm A})^{bd} (T^c_{\rm A})^{de} - (T^c_{\rm A})^{bd} (T^a_{\rm A})^{de} = i f^{acd} (T^d_{\rm A})^{be}$$

$$[T^a_{\rm A}, T^c_{\rm A}] = i f^{acd} T^d_{\rm A}$$

The index of a representation $T(\mathrm{R})$:

$$\mathrm{Tr}(T^a_\mathrm{R} T^b_\mathrm{R}) = T(\mathrm{R})\delta^{ab}$$

The quadratic Casimir $C(\mathrm{R})$:

multiplies the identity matrix

$$T^a_\mathrm{R} T^a_\mathrm{R} = C(\mathrm{R})$$

commutes with every generator, homework S-69.2

Useful relation:

$$T(\mathrm{R})D(\mathrm{A}) = C(\mathrm{R})D(\mathrm{R})$$

SU(N):

$T(\mathrm{N}) = \frac{1}{2}$

$T(\mathrm{A}) = N$

$D(\mathrm{A}) = N^2 - 1$

SO(N):

$T(\mathrm{N}) = 2$

$T(\mathrm{A}) = 2N - 4$

$D(\mathrm{A}) = \frac{1}{2}N(N-1)$

A representation is reducible if there is a unitary transformation

$$T^a_\mathrm{R} \to U^{-1} T^a_\mathrm{R} U$$

that brings all the generators to the same block diagonal form (with at least two blocks); otherwise it is irreducible.

For example, consider a reducible representation R that can put into two blocks, then R is a direct sum representation:

$$\mathrm{R} = \mathrm{R}_1 \oplus \mathrm{R}_2$$

and we have:

$$D(\mathrm{R}_1 \oplus \mathrm{R}_2) = D(\mathrm{R}_1) + D(\mathrm{R}_2)$$

$\mathrm{Tr}(T^a_\mathrm{R} T^b_\mathrm{R}) = T(\mathrm{R})\delta^{ab}$

$$T(\mathrm{R}_1 \oplus \mathrm{R}_2) = T(\mathrm{R}_1) + T(\mathrm{R}_2)$$

Consider a field that carries two group indices $\varphi_{iI}(x)$:

R_1 $\quad$ R_2

then the field is in the direct product representation:

$$\mathrm{R}_1 \otimes \mathrm{R}_2$$

The corresponding generator matrix is:

$$(T^a_{\mathrm{R}_1 \otimes \mathrm{R}_2})_{iI,jJ} = (T^a_{\mathrm{R}_1})_{ij}\delta_{IJ} + \delta_{ij}(T^a_{\mathrm{R}_2})_{IJ}$$

and we have:

$$D(\mathrm{R}_1 \otimes \mathrm{R}_2) = D(\mathrm{R}_1)D(\mathrm{R}_2)$$

$$T(\mathrm{R}_1 \otimes \mathrm{R}_2) = T(\mathrm{R}_1)D(\mathrm{R}_2) + D(\mathrm{R}_1)T(\mathrm{R}_2)$$

to prove this we use the fact that $(T^a_\mathrm{R})_{ii} = 0$.

We will use the following notation for indices of a complex representation:

$$\varphi_i \qquad\qquad i = 1, 2, \dots, D(\mathrm{R})$$

hermitian conjugation changes R to $\overline{\mathrm{R}}$ and for a field in the conjugate representation we will use the upper index

$$(\varphi_i)^\dagger = \varphi^{\dagger\, i}$$

we write generators as:

$$(T^a_\mathrm{R})_i{}^j$$

indices are contracted only if one is up and one is down!

an infinitesimal group transformation of φ_i is:

$$\varphi_i \to (1 - i\theta^a T^a_\mathrm{R})_i{}^j \varphi_j$$

$$= \varphi_i - i\theta^a (T^a_\mathrm{R})_i{}^j \varphi_j$$

generator matrices for $\overline{\mathrm{R}}$ are then given by

$$(T^a_{\overline{\mathrm{R}}})^i{}_j = -(T^a_{\mathrm{R}})_j{}^i \qquad\qquad T^a_{\overline{\mathrm{R}}} = -(T^a_{\mathrm{R}})^*$$

we trade complex conjugation for transposition

and an infinitesimal group transformation of $\varphi^{\dagger i}$ is:

$$\begin{aligned}
\varphi^{\dagger i} &\to (1 - i\theta^a T^a_{\overline{\mathrm{R}}})^i{}_j \varphi^{\dagger j} \\
&= \varphi^{\dagger i} - i\theta^a (T^a_{\overline{\mathrm{R}}})^i{}_j \varphi^{\dagger j} \\
&= \varphi^{\dagger i} + i\theta^a (T^a_{\mathrm{R}})_j{}^i \varphi^{\dagger j}
\end{aligned}$$

$(T^a_{\overline{\mathrm{R}}})^i{}_j = -(T^a_{\mathrm{R}})_j{}^i$

$$\begin{aligned}
\varphi_i &\to (1 - i\theta^a T^a_{\mathrm{R}})_i{}^j \varphi_j \\
&= \varphi_i - i\theta^a (T^a_{\mathrm{R}})_i{}^j \varphi_j
\end{aligned}$$

$$\varphi^{\dagger i}\varphi_i$$

is invariant!

Consider the Kronecker delta symbol

$\overline{\mathrm{R}}$

R

$$\begin{aligned}
\delta_i{}^j &\to (1 + i\theta^a T^a_{\mathrm{R}})_i{}^k (1 + i\theta^a T^a_{\overline{\mathrm{R}}})^j{}_l \delta_k{}^l \\
&= (1 + i\theta^a T^a_{\mathrm{R}})_i{}^k \delta_k{}^l (1 - i\theta^a T^a_{\mathrm{R}})_l{}^j \\
&= \delta_i{}^j + O(\theta^2)\,.
\end{aligned}$$

is an invariant symbol of the group!

this means that the product of the representations R and $\overline{\mathrm{R}}$ must contain the singlet representation 1, specified by $T^a_1 = 0$.

Thus we can write:

$$\mathrm{R} \otimes \overline{\mathrm{R}} = 1 \oplus \ldots$$

Another invariant symbol:

A $\quad\overline{\mathrm{R}}$

R

$$\begin{aligned}
(T^b_{\mathrm{R}})_i{}^j &\to (1 - i\theta^a T^a_{\mathrm{R}})_i{}^k (1 - i\theta^a T^a_{\overline{\mathrm{R}}})^j{}_l (1 - i\theta^a T^a_{\mathrm{A}})^{bc} (T^c_{\mathrm{R}})_k{}^l \\
&= (T^b_{\mathrm{R}})_i{}^j - i\theta^a [(T^a_{\mathrm{R}})_i{}^k (T^b_{\mathrm{R}})_k{}^j + (T^a_{\overline{\mathrm{R}}})^j{}_l (T^b_{\mathrm{R}})_i{}^l + (T^a_{\mathrm{A}})^{bc} (T^c_{\mathrm{R}})_i{}^j] \\
&\quad + O(\theta^2)\,.
\end{aligned}$$

$(T^a_{\overline{\mathrm{R}}})^i{}_j = -(T^a_{\mathrm{R}})_j{}^i \qquad (T^a_{\mathrm{A}})^{bc} = -if^{abc}$

$$\begin{aligned}
[\ldots] &= (T^a_{\mathrm{R}})_i{}^k (T^b_{\mathrm{R}})_k{}^j - (T^a_{\mathrm{R}})_l{}^j (T^b_{\mathrm{R}})_i{}^l - if^{abc} (T^c_{\mathrm{R}})_i{}^j \\
&= (T^a_{\mathrm{R}} T^b_{\mathrm{R}})_i{}^j - (T^b_{\mathrm{R}} T^a_{\mathrm{R}})_i{}^j - if^{abc} (T^c_{\mathrm{R}})_i{}^j \\
&= 0\,,
\end{aligned}$$

this implies that:

$$\mathrm{R} \otimes \overline{\mathrm{R}} \otimes \mathrm{A} = 1 \oplus \ldots$$

must contain the singlet representation!

$$\mathrm{R} \otimes \overline{\mathrm{R}} \otimes \mathrm{A} = 1 \oplus \ldots$$

$\mathrm{R} \otimes \overline{\mathrm{R}} = 1 \oplus \ldots$

$\mathrm{A} \otimes \mathrm{A} = 1 \oplus \ldots$

(A is real)

multiplying by A we find:

$$\mathrm{R} \otimes \overline{\mathrm{R}} = \mathrm{A} \oplus \ldots$$

combining it with a previous result we get

$\mathrm{R} \otimes \overline{\mathrm{R}} = 1 \oplus \ldots$

$$\mathrm{R} \otimes \overline{\mathrm{R}} = 1 \oplus \mathrm{A} \oplus \ldots$$

the product of a representation with its complex conjugate is always reducible into a sum that contains at least the singlet and the adjoint representations!

For the fundamental representation N of SU(N) we have:

$$\mathrm{N} \otimes \overline{\mathrm{N}} = 1 \oplus \mathrm{A}$$

$D(1) = 1$

$D(\mathrm{N}) = D(\overline{\mathrm{N}}) = N$

$D(\mathrm{A}) = N^2 - 1$

(no room for anything else)

Consider a real representation R :

$$\mathrm{R}\otimes\overline{\mathrm{R}} = 1\oplus \mathrm{A}\oplus\dots \qquad \overline{\mathrm{R}} = \mathrm{R}$$

$$\mathrm{R}\otimes\mathrm{R} = 1\oplus \mathrm{A}\oplus\dots$$

implies the existence of an invariant symbol with two R indices

$$\delta_{ij} \to (1-i\theta^a T^a_{\mathrm{R}})_i{}^k(1-i\theta^a T^a_{\mathrm{R}})_j{}^l\delta_{kl}$$

$$= \delta_{ij} - i\theta^a[\cancel{(T^a_{\mathrm{R}})_{ij} + (T^a_{\mathrm{R}})_{ji}}] + O(\theta^2)$$

For the fundamental representation N of SO(N) we have:

$$\mathrm{N}\otimes\mathrm{N} = 1_{\mathrm{S}}\oplus \mathrm{A}_{\mathrm{A}}\oplus \mathrm{S}_{\mathrm{S}}$$

$D(1) = 1$
$D(\mathrm{N}) = N$
$D(\mathrm{A}) = \frac{1}{2}N(N-1)$
$D(\mathrm{S}) = \frac{1}{2}N(N+1)-1$

$$\delta_{ij} = \delta_{ji}$$

corresponds to a field with a symmetric traceless pair of fundamental indices

$$\varphi_{ij} = \varphi_{ji}$$

R is pseudoreal if it is not real but there is a transformation such that $-(T^a_{\mathrm{R}})^* = V^{-1}T^a_{\mathrm{R}}V$

Consider now a pseudoreal representation:

$$\mathrm{R}\otimes\mathrm{R} = 1\oplus \mathrm{A}\oplus\dots$$

still holds but the Kronecker delta is not the corresponding invariant symbol:

$$\delta_{ij} \to (1-i\theta^a T^a_{\mathrm{R}})_i{}^k(1-i\theta^a T^a_{\mathrm{R}})_j{}^l\delta_{kl}$$

$$= \delta_{ij} - i\theta^a[(T^a_{\mathrm{R}})_{ij} + (T^a_{\mathrm{R}})_{ji}] + O(\theta^2)$$

the only alternative is to have the singlet appear in the antisymmetric part of the product. For SU(N) another invariant symbol is the Levi-Civita symbol with N indices:

$$\det U = 1$$

$$\varepsilon_{i_1\dots i_N} \to U_{i_1}{}^{j_1}\dots U_{i_N}{}^{j_N}\varepsilon_{j_1\dots j_N}$$

$$= (\det U)\varepsilon_{i_1\dots i_N}\ .$$

similarly for $\varepsilon^{i_1\dots i_N}$.

For SU(2):

$$2\otimes 2 = 1_{\mathrm{A}}\oplus 3_{\mathrm{S}}$$

we can use ε^{ij} and ε_{ij} to raise and lower SU(2) indices; if φ_i is in the 2 representation, then we can get a field in the $\overline{2}$ representation by raising the index: $\varphi^i = \varepsilon^{ij}\varphi_j$.

Another invariant symbol of interest is f^{abc} : $(T^a_{\mathrm{A}})^{bc} = -if^{abc}$

generator matrices in any rep. are invariant, or

$$T(\mathrm{R})f^{abc} = -i\,\mathrm{Tr}(T^a_{\mathrm{R}}[T^b_{\mathrm{R}}, T^c_{\mathrm{R}}])$$

the right-hand side is obviously invariant.

Very important invariant symbol is the anomaly coefficient of the rep.:

$$A(\mathrm{R})d^{abc} \equiv \tfrac{1}{2}\mathrm{Tr}(T^a_{\mathrm{R}}\{T^b_{\mathrm{R}}, T^c_{\mathrm{R}}\})$$

is completely symmetric

normalized so that $A(\mathrm{N}) = 1$ for SU(N) with $N\geq 3$.

Since $(T^a_{\overline{\mathrm{R}}})^i{}_j = -(T^a_{\mathrm{R}})_j{}^i$ we have:

$$A(\overline{\mathrm{R}}) = -A(\mathrm{R})$$

for real or pseudoreal representations $A(\mathrm{R}) = 0$.

e.g. for SU(2), all representation are real or pseudoreal and $A(\mathrm{R}) = 0$ for all of them

we also have:

$$A(\mathrm{R}_1\oplus\mathrm{R}_2) = A(\mathrm{R}_1) + A(\mathrm{R}_2)\ ,$$

$$A(\mathrm{R}_1\otimes\mathrm{R}_2) = A(\mathrm{R}_1)D(\mathrm{R}_2) + D(\mathrm{R}_1)A(\mathrm{R}_2)$$

The path integral for photons

based on S-57

We will discuss the path integral for photons and the photon propagator more carefully using the Lorentz gauge:

$$Z_0(J) = \int \mathcal{D}A\; e^{iS_0}\ ,$$

$$\mathcal{L} = +\tfrac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu A_\mu$$

as in the case of scalar field we Fourier-transform to the momentum space:

$$S_0 = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\Big[-\widetilde\varphi(k)(k^2+m^2)\widetilde\varphi(-k) + \widetilde J(k)\widetilde\varphi(-k) + \widetilde J(-k)\widetilde\varphi(k)\Big]$$

$$S_0 = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\Big[-\widetilde A_\mu(k)\Big(k^2g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde A_\nu(-k) + \widetilde J^\mu(k)\widetilde A_\mu(-k) + \widetilde J^\mu(-k)\widetilde A_\mu(k)\Big]$$

we shift integration variables so that mixed terms disappear...

$$\widetilde\chi(k) = \widetilde\varphi(k) - \frac{\widetilde J(k)}{k^2+m^2}$$

Problem: the matrix has zero eigenvalue and cannot be inverted.

To see this, note:

$$k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k)$$

where

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2$$

is a projection matrix

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k)$$

and so the only allowed eigenvalues are 0 and +1

Since

$$P^{\mu\nu}(k) k_\nu = 0$$

$$g_{\mu\nu} P^{\mu\nu}(k) = 3$$

it has one 0 and three +1 eigenvalues.

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\big(k^2 g^{\mu\nu} - k^\mu k^\nu\big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big]$$

We can decompose the gauge field $\widetilde{A}_\mu(k)$ into components aligned along a set of linearly independent four-vectors, one of which is k_μ and then this component does not contribute to the quadratic term because

$$P^{\mu\nu}(k) k_\nu = 0$$

and it doesn't even contribute to the linear term because

$$\partial^\mu J_\mu(x) = 0 \quad \longrightarrow \quad k^\mu \widetilde{J}_\mu(k) = 0$$

and so there is no reason to integrate over it; we define the path integral as integral over the remaining three basis vector; these are given by

$$k^\mu \widetilde{A}_\mu(k) = 0$$

which is equivalent to

$$\partial^\mu A_\mu(x) = 0$$

Lorentz gauge

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2$$

Within the subspace orthogonal to k_μ the projection matrix is simply the identity matrix and the inverse is straightforward; thus we get:

$$Z_0(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \widetilde{J}_\mu(k)\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\, \widetilde{J}_\nu(-k)\right]$$

$$= \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y)\right]$$

going back to the position space

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}$$

propagator in the Lorentz gauge (Landau gauge)

we can again neglect the term with momenta because the current is conserved and we obtain the propagator in the Feynman gauge:

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k) \qquad \tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon}$$

The path integral for nonabelian gauge theory

based on S-71

Now we want to evaluate the path integral for nonabelian gauge theory:

$$Z(J) \propto \int \mathcal{D}A\, e^{iS_{\rm YM}(A,J)}\ ,$$

$$S_{\rm YM}(A,J) = \int d^4x \left[-\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + J^{a\mu}A^a_\mu\right]$$

for U(1) gauge theory, the component of the gauge field parallel to the four-momentum k^μ did not appear in the action and so it should not be integrated over; since the U(1) gauge transformation is of the form $A_\mu(x) \to A_\mu(x) - \partial_\mu \Gamma(x)$, excluding the components parallel to k^μ removes the gauge redundancy in the path integral.

nonabelian gauge transformation is nonlinear:

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \tfrac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

$$A_\mu(x) = A^a_\mu(x) T^a$$

for an infinitesimal transformation:

$$U(x) = I - ig\theta(x) + O(\theta^2)$$
$$= I - ig\theta^a(x)T^a + O(\theta^2)$$

we have:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

$$A_\mu(x) \to A_\mu(x) + ig[A_\mu(x), \theta(x)] - \partial_\mu\theta(x)$$

or, in components:

$$A_\mu(x) = A_\mu^a(x)T^a$$

$$\begin{aligned} A_\mu^a(x) &\to A_\mu^a(x) - gf^{abc}A_\mu^b(x)\theta^c(x) - \partial_\mu\theta^a(x) \\ &= A_\mu^a(x) - [\delta^{ac}\partial_\mu + gf^{abc}A_\mu^b(x)]\theta^c(x) \\ &= A_\mu^a(x) - [\delta^{ac}\partial_\mu - igA_\mu^b(-if^{bac})]\theta^c(x) \\ &= A_\mu^a(x) - [\delta^{ac}\partial_\mu - igA_\mu^b(T_A^b)^{ac}]\theta^c(x) \\ &= A_\mu^a(x) - D_\mu^{ac}\theta^c(x)\ , \end{aligned}$$

the covariant derivative in the adjoint representation
(instead of ∂_μ that we have for the U(1) transformation)

we have to remove the gauge redundancy in a different way!

Consider an ordinary integral of the form:

$$Z \propto \int dx\, dy\, e^{iS(x)}$$

the integral over y is redundant
we can simply drop it and define:

$$Z \equiv \int dx\, e^{iS(x)}$$

this is how we dealt with gauge redundancy in the abelian case

or we can get the same result by inserting a delta function:

$$Z = \int dx\, dy\, \delta(y)\, e^{iS(x)}$$

this is what we are going to do for the nonabelian case

the argument of the delta function can be shifted by an arbitrary function of x

$$Z = \int dx\, dy\, \delta(y - f(x))\, e^{iS(x)}$$

$$Z = \int dx\, dy\, \delta(y - f(x))\, e^{iS(x)}$$

if $y = f(x)$ is a unique solution of $G(x,y) = 0$ for fixed x, we can write:

$$\delta(G(x,y)) = \frac{\delta(y - f(x))}{|\partial G/\partial y|}$$

then we have:

$$Z = \int dx\, dy\, \frac{\partial G}{\partial y}\, \delta(G)\, e^{iS}$$

we dropped the abs. value

generalizing the result to an integral over n variables:

$$Z = \int d^n x\, d^n y\, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i)\, e^{iS}$$

Now we translate this result to path integral over nonabelian gauge fields:

$$Z = \int d^n x\, d^n y\, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i)\, e^{iS}$$

i index now represents x and a

x and y $\longrightarrow$ $A_\mu^a(x)$

y $\longrightarrow$ $\theta^a(x)$

G becomes the gauge fixing function:
for R_ξ gauge we use:

$$G^a(x) \equiv \partial^\mu A_\mu^a(x) - \omega^a(x)$$

fixed, arbitrarily chosen function of x

$$Z(J) \propto \int \mathcal{D}A\, \det\left(\frac{\delta G}{\delta\theta}\right) \prod_{x,a} \delta(G)\, e^{iS_{\rm YM}}$$

$$S_{\rm YM}(A,J) = \int d^4x \left[-\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + J^{a\mu}A^a_\mu\right]$$

let's evaluate the functional derivative:

$$Z(J) \propto \int \mathcal{D}A \, \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G)\, e^{iS_{\rm YM}}$$

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$
$$A^a_\mu(x) \to A^a_\mu(x) - D^{ac}_\mu \theta^c(x)$$

$$\longrightarrow \quad G^a(x) \to G^a(x) - \partial^\mu D^{ab}_\mu \theta^b(x)$$

and we find:

$$\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} = \delta_{ba}\delta^4(y-x)$$

$$\frac{\delta G^a(x)}{\delta \theta^b(y)} = -\partial^\mu D^{ab}_\mu \delta^4(x-y)$$

Recall, the functional determinant can be written as a path integral over complex Grassmann variables:

$$\int d^n\bar\psi \, d^n\psi \, \exp\left(-i\bar\psi_i M_{ij}\psi_j\right) \propto \det M$$

$$\det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int \mathcal{D}c\,\mathcal{D}\bar c\; e^{iS_{\rm gh}} \qquad S_{\rm gh} = \int d^4x\, \mathcal{L}_{\rm gh}$$

where:

$$\mathcal{L}_{\rm gh} = \bar c^a \partial^\mu D^{ab}_\mu c^b$$

Faddeev-Popov ghosts

the ghost lagrangian can be further written as:

$$\begin{aligned}\mathcal{L}_{\rm gh} &= \bar c^a \partial^\mu D^{ab}_\mu c^b \\ &= -\partial^\mu \bar c^a D^{ab}_\mu c^b \\ &= -\partial^\mu \bar c^a \partial_\mu c^a + ig\partial^\mu \bar c^a A^c_\mu (T^c_{\rm A})^{ab} c^b \\ &= \underline{-\partial^\mu \bar c^a \partial_\mu c^a + g f^{abc} A^c_\mu \partial^\mu \bar c^a c^b}\,.\end{aligned}$$

we drop the total divergence

$$D^{ac}_\mu = \delta^{ac}\partial_\mu - igA^b_\mu (T^b_{\rm A})^{ac}$$

Comments:

- ghost fields interact with the gauge field; however ghosts do not exist and we will see later (when we discuss the BRST symmetry) that the amplitude to produce them in any scattering process is zero. The only place they appear is in loops! Since they are Grassmann fields, a closed loop of ghost lines in a Feynman diagram comes with a minus sign!
- For abelian gauge theory $f^{abc} = 0$ and thus there is no interaction term for ghost fields; we can absorb its path integral into overall normalization.

At this point we have:

$$Z(J) \propto \int \mathcal{D}A \, \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G)\, e^{iS_{\rm YM}}$$

$$\det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int \mathcal{D}c\,\mathcal{D}\bar c\; e^{iS_{\rm gh}} \qquad G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$

fixed, arbitrarily chosen function of x

The path integral is independent of $\omega^a(x)$! Thus we can multiply it by arbitrary functional of ω and perform a path integral over ω; the result changes only the overall normalization of $Z(J)$.

we can multiply $Z(J)$ by:

$$\exp\left[-\frac{i}{2\xi}\int d^4x\, \omega^a\omega^a\right]$$

integral over ω is trivial

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu$$

gauge fixing term

$$S_{\rm gf} = \int d^4x\, \mathcal{L}_{\rm gf}$$

our final result is:

$$Z(J) \propto \int \mathcal{D}A\,\mathcal{D}\bar c\,\mathcal{D}c\; \exp\left(iS_{\rm YM} + iS_{\rm gh} + iS_{\rm gf}\right)$$

next time we will derive Feynman rules from this action...

The Feynman rules for nonabelian gauge theory

based on S-72

The lagrangian for nonabelian gauge theory is:

$$\begin{aligned}\mathcal{L}_{\rm YM} &= -\tfrac14 F^{e\mu\nu}F^e_{\mu\nu} \\ &= -\tfrac14(\partial^\mu A^{e\nu} - \partial^\nu A^{e\mu} + gf^{abe}A^{a\mu}A^{b\nu})(\partial_\mu A^e_\nu - \partial_\nu A^e_\mu + gf^{cde}A^c_\mu A^d_\nu) \\ &= -\tfrac12 \partial^\mu A^{e\nu}\partial_\mu A^e_\nu + \tfrac12 \partial^\mu A^{e\nu}\partial_\nu A^e_\mu \\ &\quad - gf^{abe}A^{a\mu}A^{b\nu}\partial_\mu A^e_\nu - \tfrac14 g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\,.\end{aligned}$$

the gauge fixing term for R_ξ gauge:

$$\mathcal{L}_{\rm gf} = -\tfrac12 \xi^{-1}\partial^\mu A^e_\mu \partial^\nu A^e_\nu$$

we can write the gauge fixed lagrangian in the form:

$$\begin{aligned}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} &= \tfrac12 A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{e\nu} + \tfrac12\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} \\ &\quad - gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac14 g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\end{aligned}$$

The gluon propagator in the R_ξ gauge:

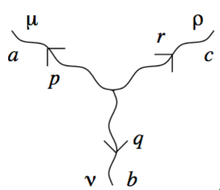
$$\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{gf}} = \underline{\tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^{e\nu}+\tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu}} \\ - g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu$$

going to the momentum space and taking
the inverse of the quadratic term

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2-i\epsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}+\xi\frac{k_\mu k_\nu}{k^2}\right)$$

The three-gluon vertex:

$$\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{gf}} = \tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^{e\nu}+\tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} \\ \underline{- g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu} - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu$$



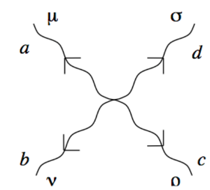
all photons outgoing

the derivative acting on an outgoing particle
brings (-i momentum) of the particle

$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) = i(-gf^{abc})(-ir_\mu g_{\nu\rho}) \\ +[\,5 \text{ permutations of } (a,\mu,p),(b,\nu,q),(c,\rho,r)\,] \\ = gf^{abc}[(q-r)_\mu g_{\nu\rho}+(r-p)_\nu g_{\rho\mu}+(p-q)_\rho g_{\mu\nu}]\,.$$

The four-gluon vertex:

$$\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{gf}} = \tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^{e\nu}+\tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} \\ - g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu \underline{- \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu}$$



$$i\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} = -ig^2 f^{abe}f^{cde}g_{\mu\rho}g_{\nu\sigma} \\ +[\,5 \text{ permutations of } (b,\nu),(c,\rho),(d,\sigma)\,] \\ = -ig^2\,[\,f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\ + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}) \\ + f^{ade}f^{bce}(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\rho}g_{\sigma\nu})\,]\,.$$

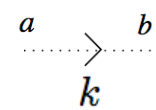
The ghost lagrangian:

we need ghosts for loop calculations

$$\mathcal{L}_{\mathrm{gh}} = -\partial^\mu\bar{c}^b D^{bc}_\mu c^c \\ = -\partial^\mu\bar{c}^c\partial_\mu c^c + ig\partial^\mu\bar{c}^b A^a_\mu (T^a_{\mathrm{A}})^{bc}c^c \\ = \underline{-\partial^\mu\bar{c}^c\partial_\mu c^c} + gf^{abc}A^a_\mu\partial^\mu\bar{c}^b c^c\,.$$

The ghost propagator:

massless complex scalar
they carry charge arrow
(and also a group index)



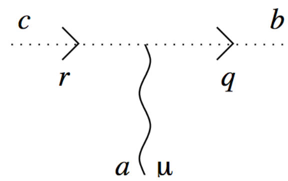
$$\tilde{\Delta}^{ab}(k^2) = \frac{\delta^{ab}}{k^2-i\epsilon}$$

The ghost-ghost-gluon vertex:

$$\mathcal{L}_{\rm gh} = -\partial^\mu \bar{c}^c \partial_\mu c^c + \underline{g f^{abc} A^a_\mu \partial^\mu \bar{c}^b c^c}$$

the derivative acting on an outgoing particle brings (-i momentum) of the particle

ghosts are complex scalars so their propagator carry a charge arrow



$$i\mathbf{V}^{abc}_\mu(q,r) = i(gf^{abc})(-iq_\mu)$$

$$= gf^{abc} q_\mu \,.$$

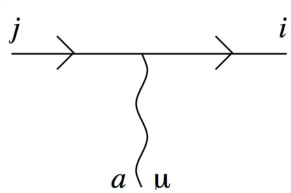
Finally we can include quarks:

$$\mathcal{L}_{\rm q} = i\overline{\Psi}_i \not{D}_{ij} \Psi_j - m\overline{\Psi}_i \Psi_i$$

$$= i\overline{\Psi}_i \not\partial \Psi_i - m\overline{\Psi}_i\Psi_i + \underline{gA^a_\mu \overline{\Psi}_i \gamma^\mu T^a_{ij}\Psi_j}$$

propagator:

$$\tilde{S}_{ij}(p) = \frac{(-\not{p}+m)\delta_{ij}}{p^2+m^2-i\epsilon}$$



vertex:

$$i\mathbf{V}^{\mu a}_{ij} = ig\gamma^\mu T^a_{ij}$$

for fields in different representations we would have $(T^a_{\rm R})_{ij}$.

The beta function in nonabelian gauge theory

based on S-73

The complete (renormalized) lagrangian for nonabelian gauge theory is:

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2} Z_3 A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} + \tfrac{1}{2}\xi^{-1}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} \\ & - Z_{3g} g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \tfrac{1}{4} Z_{4g} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu \\ & - Z_{2'} \partial^\mu \bar{C}^a \partial_\mu C^a + Z_{1'} g f^{abc} A^c_\mu \partial^\mu \bar{C}^a C^b \\ & + iZ_2 \overline{\Psi}_i \not\partial \Psi_i - Z_m m \overline{\Psi}_i \Psi_i + Z_1 g A^a_\mu \overline{\Psi}_i \gamma^\mu T^a_{ij} \Psi_j \,.\end{aligned}$$

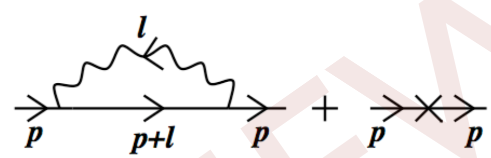
from gauge invariance we expect:

$d = 4 - \varepsilon$

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^\varepsilon$$

Slavnov-Taylor identities
(non-abelian analogs of Ward identities)

There is only one diagram contributing at one loop level:



$$i\Sigma(\not{p}) = (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \tilde{S}(\not{p}+\not{\ell})\gamma^\mu\right] \tilde{\Delta}_{\mu\nu}(\ell)$$

$$- i(Z_2 - 1)\not{p} - i(Z_m - 1)m + O(e^4) \,.$$

the photon propagator in the Feynman gauge:

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2 + m_\gamma^2 - i\epsilon}$$

fictitious photon mass

following the usual procedure:

$$i\Sigma(\not p) = (iZ_1 e)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\left[\gamma^\nu \tilde S(\not p+\not\ell)\gamma^\mu\right]\tilde\Delta_{\mu\nu}(\ell)$$
$$-\,i(Z_2-1)\not p - i(Z_m-1)m + O(e^4)\ .$$

$$i\Sigma(\not p) = e^2\tilde\mu^\varepsilon\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\,\frac{N}{(q^2+D)^2}$$
$$-\,i(Z_2-1)\not p - i(Z_m-1)m + O(e^4)$$

$$q = \ell + xp$$
$$D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2\ ,$$
$$N = \gamma_\mu(-\not p - \not\ell + m)\gamma^\mu$$

$$\gamma_\mu\gamma^\mu = -d \qquad \gamma_\mu\not p\gamma^\mu = (d-2)\not p$$

$$= -(d-2)(\not p+\not\ell) - dm$$
$$= -(d-2)[\cancel{\not q} + (1-x)\not p] - dm\ ,$$

we get:

$$\Sigma(\not p) = -\frac{e^2}{8\pi^2}\int_0^1 dx\Big((2-\varepsilon)(1-x)\not p + (4-\varepsilon)m\Big)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right]$$
$$-\,(Z_2-1)\not p - (Z_m-1)m + O(e^4)\ .$$

$$\Sigma(\not p) = -\frac{e^2}{8\pi^2}\int_0^1 dx\Big((2-\varepsilon)(1-x)\not p + (4-\varepsilon)m\Big)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right]$$
$$-\,(Z_2-1)\not p - (Z_m-1)m + O(e^4)\ .$$

we set Z's to cancel divergent parts

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)$$
$$Z_m = 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)$$

we can impose $\Sigma(-m) = 0$ by writing:

$$\Sigma(\not p) = \frac{e^2}{8\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not p + 2m\Big)\ln(D/D_0) + \kappa_2(\not p + m)\right] + O(e^4)$$

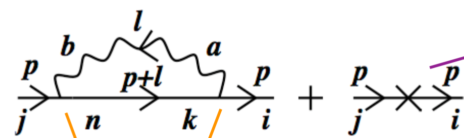
$$D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$$
$$D_0 = x^2m^2 + (1-x)m_\gamma^2$$

fixed by imposing: $\Sigma'(-m) = 0$

$$\kappa_2 = -2\int_0^1 dx\,x(1-x^2)m^2/D_0$$
$$= -2\ln(m/m_\gamma) + 1\ ,$$

we work in Feynman gauge and use the $\overline{\text{MS}}$ scheme

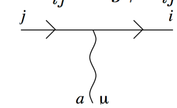
Let's start with the quark propagator:



$$-(Z_2-1)\delta_{ij}\not p$$

the calculation is identical to QED with additional color factor:

$$i\mathbf{V}^{\mu a}_{ij} = ig\gamma^\mu T^a_{ij}$$

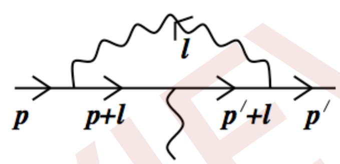


$$(T^aT^a)_{ij} = C(\mathrm{R})\delta_{ij}$$

the result has to be identical to QED up to the color factor:

$$Z_2 = 1 - C(\mathrm{R})\,\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)$$

Finally, let's evaluate the diagram contributing to the vertex:



$$q \equiv p' - p$$

$$i\mathbf{V}^\mu(p',p) = iZ_1e\gamma^\mu + i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) + O(e^5)$$

$$i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = (ie)^3\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho\tilde S(\not p'+\not\ell)\gamma^\mu\tilde S(\not p+\not\ell)\gamma^\nu\right]\tilde\Delta_{\nu\rho}(\ell)$$

$$\tilde\Delta_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2 + m_\gamma^2 - i\epsilon}$$

combining denominators...

$$i\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p) = (ie)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho \tilde{S}(\not{p}'+\not{\ell})\gamma^\mu \tilde{S}(\not{p}+\not{\ell})\gamma^\nu\right]\tilde{\Delta}_{\nu\rho}(\ell)$$

$$i\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p) = e^3 \int dF_3 \int \frac{d^4q}{(2\pi)^4}\,\frac{N^\mu}{(q^2+D)^3}$$

$$\int dF_3 \equiv 2\int_0^1 dx_1 dx_2 dx_3\, \delta(x_1+x_2+x_3-1)$$

$$\begin{aligned}
q &= \ell + x_1 p + x_2 p' ,\\
D &= x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2 p\cdot p'\\
&\quad + (x_1+x_2)m^2 + x_3 m_\gamma^2 ,\\
N^\mu &= \gamma_\nu(-\not{p}' - \not{\ell} + m)\gamma^\mu(-\not{p} - \not{\ell} + m)\gamma^\nu\\
&= \gamma_\nu[-\not{q} + x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-\not{q} - (1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu\\
&= \gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu + \tilde{N}^\mu + (\text{linear in } q)
\end{aligned}$$

$$\tilde{N}^\mu = \gamma_\nu[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu$$

$$i\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p) = e^3 \int dF_3 \int \frac{d^4q}{(2\pi)^4}\,\frac{N^\mu}{(q^2+D)^3}$$

$$N^\mu = \gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu + \tilde{N}^\mu + (\text{linear in } q)$$

$$\tilde{N}^\mu = \gamma_\nu[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu$$

continuing to d dimensions

$$\gamma_\rho\gamma^\mu\gamma^\rho = (d-2)\gamma^\mu$$

$$\gamma_\nu \not{q}\gamma^\mu \not{q}\gamma^\nu \to \frac{1}{d}q^2\,\gamma_\nu\gamma_\rho\gamma^\mu\gamma^\rho\gamma^\nu = \frac{(d-2)^2}{d}\,q^2\,\gamma^\mu$$

evaluating the loop integral we get:

$$\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p) = \frac{e^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - 1 - \tfrac{1}{2}\int dF_3\, \ln(D/\mu^2)\right)\gamma^\mu + \tfrac{1}{4}\int dF_3\,\frac{\tilde{N}^\mu}{D}\right]$$

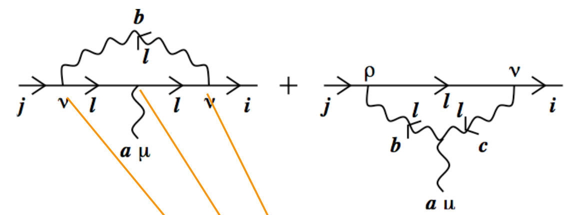
$$i\mathbf{V}^\mu(p',p) = iZ_1 e\gamma^\mu + i\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p) + O(e^5)$$

the infinite part can be absorbed by Z

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)$$

the finite part of the vertex function is fixed by a suitable condition.

Let's continue with the quark-quark-gluon vertex:



the calculation of the 1st diagram is identical to QED with additional color factor:

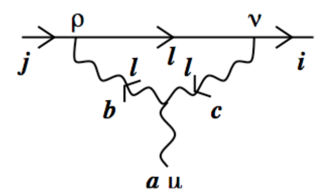
$$(T^bT^aT^b)_{ij}$$

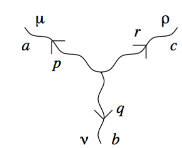
$$\begin{aligned}
T^bT^aT^b &= T^b\left(T^bT^a + if^{abc}T^c\right)\\
&= C(\mathrm{R})T^a + \tfrac{1}{2}if^{abc}[T^b,T^c]\\
&= C(\mathrm{R})T^a + \tfrac{1}{2}(if^{abc})(if^{bcd})T^d\\
&= C(\mathrm{R})T^a - \tfrac{1}{2}(T^a_{\mathrm{A}})^{bc}(T^d_{\mathrm{A}})^{cb}T^d\\
&= \left[C(\mathrm{R}) - \tfrac{1}{2}T(\mathrm{A})\right]T^a .
\end{aligned}$$

thus the divergent part is the same as in QED up to the color factor:

$$\left[C(\mathrm{R}) - \tfrac{1}{2}T(\mathrm{A})\right]\frac{g^2}{8\pi^2\varepsilon}\,igT^a_{ij}\gamma^\mu$$

Let's evaluate the second diagram:





$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) = gf^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]$$

the divergent piece doesn't depend on external momenta, so we can set them to 0; using Feynman rules we get:

$$\begin{aligned}
i\mathbf{V}^{a\mu}_{ij}(0,0) &= (ig)^2 g f^{abc}(T^cT^b)_{ij}\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\gamma_\rho(-\not{\ell}+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)}\\
&\quad\times[(\ell-(-\ell))^\mu g^{\nu\rho} + (-\ell-0)^\nu g^{\rho\mu} + (0-\ell)^\rho g^{\mu\nu}]
\end{aligned}$$

$$i\mathbf{V}^{a\mu}_{ij}(0,0) = (ig)^2 g f^{abc}(T^cT^b)_{ij}\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\gamma_\rho(-\not\ell+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)}$$
$$\times\left[(\ell-(-\ell))^\mu g^{\nu\rho}+(-\ell-0)^\nu g^{\rho\mu}+(0-\ell)^\rho g^{\mu\nu}\right]$$

$$\begin{aligned} f^{abc}T^cT^b &= \tfrac12 f^{abc}[T^c,T^b] \\ &= \tfrac12 i f^{abc}f^{cbd}T^d \\ &= -\tfrac12 i T(\mathrm{A})T^a \end{aligned}$$

the numerator is:

$$N^\mu = \gamma_\rho(-\gamma_\sigma\ell^\sigma+m)\gamma_\nu(2\ell^\mu g^{\nu\rho}-\ell^\nu g^{\rho\mu}-\ell^\rho g^{\mu\nu})$$

we can drop linear terms

$$\ell^\sigma\ell^\mu \;\to\; d^{-1}\ell^2 g^{\sigma\mu}$$

$$\begin{aligned} N^\mu &\to -d^{-1}\ell^2\,(\gamma_\rho\gamma_\sigma\gamma_\nu)(2g^{\sigma\mu}g^{\nu\rho}-g^{\sigma\nu}g^{\rho\mu}-g^{\sigma\rho}g^{\mu\nu}) \\ &\to -d^{-1}\ell^2\,(2\gamma^\nu\gamma^\mu\gamma_\nu-\gamma^\mu\gamma^\nu\gamma_\nu-\gamma_\rho\gamma^\rho\gamma^\mu) \\ &\to -d^{-1}\ell^2\,(2(d-2)+d+d)\gamma^\mu\,. \end{aligned}$$

for $d=4$ (we are interested in the divergent part only):

$$N^\mu \to -3\ell^2\gamma^\mu$$

$$i\mathbf{V}^{a\mu}_{ij}(0,0) = (ig)^2 g f^{abc}(T^cT^b)_{ij}\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\gamma_\rho(-\not\ell+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)}$$
$$\times\left[(\ell-(-\ell))^\mu g^{\nu\rho}+(-\ell-0)^\nu g^{\rho\mu}+(0-\ell)^\rho g^{\mu\nu}\right]$$

$$\begin{aligned} f^{abc}T^cT^b &= \tfrac12 f^{abc}[T^c,T^b] \\ &= \tfrac12 i f^{abc}f^{cbd}T^d \\ &= -\tfrac12 i T(\mathrm{A})T^a \end{aligned}$$

$$N^\mu \to -3\ell^2\gamma^\mu$$

thus for the divergent part of the 2nd diagram we find:

$$i\mathbf{V}^{a\mu}_{ij}(0,0) = \tfrac32 T(\mathrm{A})\,g^3\,T^a_{ij}\gamma^\mu\underbrace{\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2(\ell^2+m^2)}}_{i/8\pi^2\varepsilon\,+\,O(\varepsilon^0)}$$

putting pieces together we get:

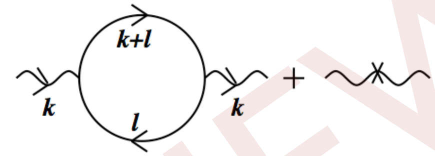
$$\mathbf{V}^{a\mu}_{ij}(0,0)_{\text{div}} = \left(Z_1+\left[C(\mathrm{R})-\tfrac12 T(\mathrm{A})\right]\frac{g^2}{8\pi^2\varepsilon}+\tfrac32 T(\mathrm{A})\,\frac{g^2}{8\pi^2\varepsilon}\right)gT^a_{ij}\gamma^\mu$$

and we find:

$$Z_1 = 1-\left[C(\mathrm{R})+T(\mathrm{A})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon}+O(g^4)$$

in Feynman gauge and the $\overline{\text{MS}}$ scheme

Let's now calculate the $i\Pi^{\mu\nu}(k)$ at one loop:



extra -1 for fermion loop; and the trace

$$\tilde S(\not p) = \frac{-\not p+m}{p^2+m^2-i\epsilon}$$

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2\left(\tfrac1i\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\tilde S(\not\ell+\not k)\gamma^\mu\tilde S(\not\ell)\gamma^\nu\right]$$
$$-\,i(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu)+O(e^4)\,,$$

$$-\tfrac14 F^{\mu\nu}F_{\mu\nu} = -\tfrac12\partial^\mu A^\nu\partial_\mu A_\nu+\tfrac12\partial^\mu A^\nu\partial_\nu A_\mu$$

$$\mathcal{L}_{\text{ct}} = i(Z_2-1)\overline\Psi\not\partial\Psi-(Z_m-1)m\overline\Psi\Psi-\tfrac14(Z_3-1)F^{\mu\nu}F_{\mu\nu}$$

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2\left(\tfrac1i\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\tilde S(\not\ell+\not k)\gamma^\mu\tilde S(\not\ell)\gamma^\nu\right]$$
$$-\,i(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu)+O(e^4)\,,$$

$$\tilde S(\not p) = \frac{-\not p+m}{p^2+m^2-i\epsilon}$$

$$\mathrm{Tr}\left[\tilde S(\not\ell+\not k)\gamma^\mu\tilde S(\not\ell)\gamma^\nu\right] = \int_0^1 dx\,\frac{4N^{\mu\nu}}{(q^2+D)^2}$$

$$q = \ell+xk$$
$$D = x(1-x)k^2+m^2-i\epsilon$$
$$4N^{\mu\nu} = \mathrm{Tr}\left[(-\not\ell-\not k+m)\gamma^\mu(-\not\ell+m)\gamma^\nu\right]$$

$$\mathrm{Tr}[\not a\not b\not c\not d] = 4\left[(ad)(bc)-(ac)(bd)+(ab)(cd)\right]$$
$$\mathrm{Tr}[\not a\not b] = -4(ab)$$

$$N^{\mu\nu} = (\ell+k)^\mu\ell^\nu+\ell^\mu(\ell+k)^\nu-[\ell(\ell+k)+m^2]g^{\mu\nu}$$

$$\ell = q-xk$$

$$N^{\mu\nu} \to 2q^\mu q^\nu-2x(1-x)k^\mu k^\nu-[q^2-x(1-x)k^2+m^2]g^{\mu\nu}$$

we ignore terms linear in q

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1 e)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde S(\not\ell+\not k)\gamma^\mu \tilde S(\not\ell)\gamma^\nu\Big]$$
$$- i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4)\ ,$$

$$\mathrm{Tr}\Big[\tilde S(\not\ell+\not k)\gamma^\mu \tilde S(\not\ell)\gamma^\nu\Big] = \int_0^1 dx\,\frac{4N^{\mu\nu}}{(q^2+D)^2}$$

$$N^{\mu\nu} \to 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu}$$

the integral diverges in 4 spacetime dimensions and so we analytically continue it to $d = 4-\varepsilon$; we also make the replacement $e \to e\tilde\mu^{\varepsilon/2}$ to keep the coupling dimensionless:

$$\int d^dq\, q^\mu q^\nu f(q^2) = \frac{1}{d}\, g^{\mu\nu} \int d^dq\, q^2 f(q^2)$$

see your homework

$$N^{\mu\nu} \to -2x(1-x)k^\mu k^\nu + \Big[\Big(\tfrac{2}{d}-1\Big)q^2 + x(1-x)k^2 - m^2\Big]g^{\mu\nu}$$

$$(\tfrac{2}{d}-1)q^2 \to D$$

$$D = x(1-x)k^2 + m^2 - i\epsilon$$

$$\int \frac{d^d\bar q}{(2\pi)^d}\frac{(\bar q^2)^a}{(\bar q^2+D)^b} = \frac{\Gamma(b-a-\frac12 d)\Gamma(a+\frac12 d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac12 d)}\,D^{-(b-a-d/2)}$$

$$(\tfrac{2}{d}-1)\int \frac{d^dq}{(2\pi)^d}\frac{q^2}{(q^2+D)^2} = D\int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+D)^2}$$

$$N^{\mu\nu} \to 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

is transverse :)

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1 e)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde S(\not\ell+\not k)\gamma^\mu \tilde S(\not\ell)\gamma^\nu\Big]$$
$$- i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4)\ ,$$

$$\mathrm{Tr}\Big[\tilde S(\not\ell+\not k)\gamma^\mu \tilde S(\not\ell)\gamma^\nu\Big] = \int_0^1 dx\,\frac{4N^{\mu\nu}}{(q^2+D)^2}$$

$$N^{\mu\nu} \to 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

the integral over q is straightforward:

$$\tilde\mu^\varepsilon \int \frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+D)^2} = \frac{i}{16\pi^2}\Gamma(\tfrac{\varepsilon}{2})\Big(4\pi\tilde\mu^2/D\Big)^{\varepsilon/2}$$
$$= \frac{i}{8\pi^2}\Big[\frac{1}{\varepsilon} - \tfrac12 \ln(D/\mu^2)\Big]$$

$$\mu^2 = 4\pi e^{-\gamma}\tilde\mu^2$$

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\Big[\frac{1}{\varepsilon} - \tfrac12\ln(D/\mu^2)\Big] - (Z_3-1) + O(e^4)$$

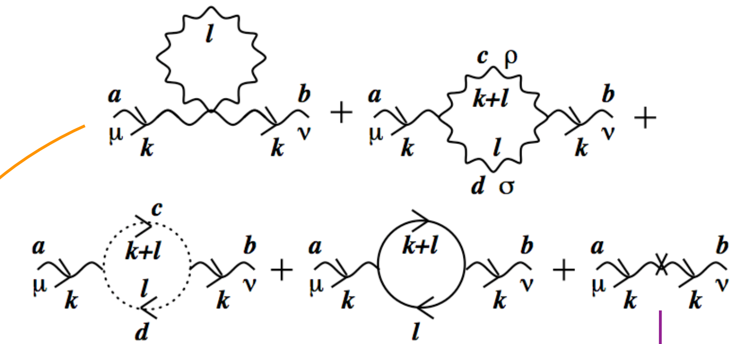
imposing $\Pi(0) = 0$ fixes

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\Big[\frac{1}{\varepsilon} - \ln(m/\mu)\Big] + O(e^4)$$

and

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln(D/m^2) + O(e^4)$$

Let's now evaluate one-loop corrections to the gluon propagator:

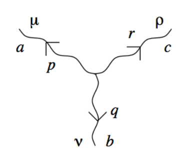


$$\tilde\mu^\varepsilon \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2} = -\frac{i}{8\pi^2}\frac{1}{\varepsilon}m^2 + O(\varepsilon^0)$$

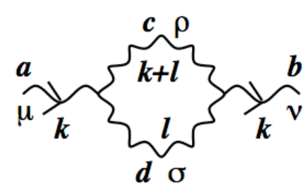
$m^2 \to 0$

0

$$-\,i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu)\,\delta^{ab}$$



$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) = g f^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]$$



$$i\Pi^{\mu\nu ab}(k) = \tfrac12 g^2 f^{acd}f^{bcd}\left(\tfrac1i\right)^2\int\frac{d^4\ell}{(2\pi)^4}\frac{N^{\mu\nu}}{\ell^2(\ell+k)^2}$$

$$N^{\mu\nu} = [((k+\ell)-(-\ell))^\mu g^{\rho\sigma} + (-\ell-(-k))^\rho g^{\sigma\mu} + ((-k)-(k+\ell))^\sigma g^{\mu\rho}]$$
$$\times[(-k-\ell)-\ell)^\nu g_{\rho\sigma} + (\ell-k)_\rho \delta_\sigma{}^\nu + (k-(-k-\ell))_\sigma \delta^\nu{}_\rho]$$
$$= -[(2\ell+k)^\mu g^{\rho\sigma} - (\ell-k)^\rho g^{\sigma\mu} - (\ell+2k)^\sigma g^{\mu\rho}]$$
$$\times[(2\ell+k)^\nu g_{\rho\sigma} - (\ell-k)_\rho \delta_\sigma{}^\nu - (\ell+2k)_\sigma \delta^\nu{}_\rho]$$

$$i\Pi^{\mu\nu ab}(k) \;=\; \tfrac{1}{2}g^2 f^{acd}f^{bcd}\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{N^{\mu\nu}}{\ell^2(\ell+k)^2}$$

$$f^{acd}f^{bcd} = T(\mathrm{A})\delta^{ab}$$

$$N^{\mu\nu} = -[(2\ell+k)^\mu g^{\rho\sigma} - (\ell-k)^\rho g^{\sigma\mu} - (\ell+2k)^\sigma g^{\mu\rho}] \times[(2\ell+k)^\nu g_{\rho\sigma} - (\ell-k)_\rho \delta_\sigma{}^\nu - (\ell+2k)_\sigma \delta^\nu{}_\rho]$$

combining denominators and continuing to $d=4-\varepsilon$ dimension:

$$-\tfrac{1}{2}g^2\,T(\mathrm{A})\delta^{ab}\,\tilde\mu^\varepsilon\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\,\frac{N^{\mu\nu}}{(q^2+D)^2}$$

$$q = \ell + xk$$
$$D = x(1-x)k^2$$

where

$$N^{\mu\nu} = -[(2q+(1-2x)k)^\mu g^{\rho\sigma} - (q-(1+x)k)^\rho g^{\sigma\mu} - (q+(2-x)k)^\sigma g^{\mu\rho}] \times[(2q+(1-2x)k)^\nu g_{\rho\sigma} - (q-(1+x)k)_\rho\delta_\sigma{}^\nu - (q+(2-x)k)_\sigma\delta^\nu{}_\rho]$$

$$N^{\mu\nu} = -[(2q+(1-2x)k)^\mu g^{\rho\sigma} - (q-(1+x)k)^\rho g^{\sigma\mu} - (q+(2-x)k)^\sigma g^{\mu\rho}] \times[(2q+(1-2x)k)^\nu g_{\rho\sigma} - (q-(1+x)k)_\rho\delta_\sigma{}^\nu - (q+(2-x)k)_\sigma\delta^\nu{}_\rho]$$

expanding and ignoring terms linear in q

$$\begin{aligned}N^{\mu\nu} \;\to\; & -2q^2 g^{\mu\nu} - (4d-6)q^\mu q^\nu \\ & -[(1+x)^2+(2-x)^2]k^2 g^{\mu\nu} \\ & -[d(1-2x)^2 + 2(1-2x)(1+x) \\ & \quad -2(2-x)(1+x) - 2(2-x)(1-2x)]k^\mu k^\nu\end{aligned}$$

we are interested in the divergent part only

$$d=4$$
$$q^\mu q^\nu \to \tfrac{1}{4}q^2 g^{\mu\nu}$$

$$N^{\mu\nu} \to -\tfrac{9}{2}q^2 g^{\mu\nu} - (5-2x+2x^2)k^2 g^{\mu\nu} + (2+10x-10x^2)k^\mu k^\nu$$

$$-\tfrac{1}{2}g^2\,T(\mathrm{A})\delta^{ab}\,\tilde\mu^\varepsilon\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\,\frac{N^{\mu\nu}}{(q^2+D)^2}$$

$$q = \ell + xk$$
$$D = x(1-x)k^2$$

$$N^{\mu\nu} \to -\tfrac{9}{2}q^2 g^{\mu\nu} - (5-2x+2x^2)k^2 g^{\mu\nu} + (2+10x-10x^2)k^\mu k^\nu$$

$$q^2 \longrightarrow \left(\tfrac{2}{d}-1\right)^{-1}D$$

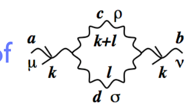
$$N^{\mu\nu} \to -(5-11x+11x^2)k^2 g^{\mu\nu} + (2+10x-10x^2)k^\mu k^\nu$$

$$\tilde\mu^\varepsilon\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2\varepsilon} + O(\varepsilon^0)$$

we get:

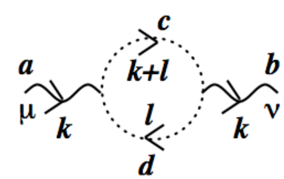
$$-\frac{ig^2}{16\pi^2}\,T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\int_0^1 dx\,N^{\mu\nu} + O(\varepsilon^0)$$

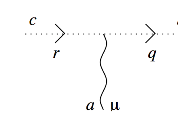
integrating over x, we get the result for the divergent part of



$$-\frac{ig^2}{16\pi^2}\,T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{19}{6}k^2 g^{\mu\nu} + \tfrac{11}{3}k^\mu k^\nu\right)$$

now let's calculate the ghost loop:





$$i\mathbf{V}^{abc}_\mu(q,r) = i(gf^{abc})(-iq_\mu)$$
$$= gf^{abc}q_\mu\,.$$

extra -1 for closed ghost loop

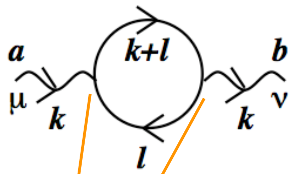
$$i\Pi^{\mu\nu ab}(k) \;=\; (-1)g^2 f^{acd}f^{bdc}\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{(\ell+k)^\mu\ell^\nu}{\ell^2(\ell+k)^2}$$

$$f^{acd}f^{bdc} = -T(\mathrm{A})\delta^{ab}$$

combining denominators, shifting the numerator, performing integrals, ...

$$-\frac{ig^2}{8\pi^2}\,T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{1}{12}k^2 g^{\mu\nu} - \tfrac{1}{6}k^\mu k^\nu\right)$$

finally, let's calculate the fermion loop:



the same calculation as in QED, except for the color factor:

$$\mathrm{Tr}(T^a T^b) = T(\mathrm{R})\delta^{ab}$$

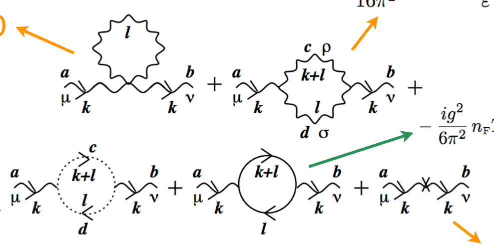
$$-\frac{ig^2}{6\pi^2} n_{\mathrm{F}} T(\mathrm{R})\delta^{ab}\,\frac{1}{\varepsilon}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$

the number of flavors

Putting pieces together:

$$-\frac{ig^2}{16\pi^2} T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{19}{6}k^2 g^{\mu\nu} + \tfrac{11}{3}k^\mu k^\nu\right)$$

0



$$-\frac{ig^2}{6\pi^2} n_{\mathrm{F}} T(\mathrm{R})\delta^{ab}\,\frac{1}{\varepsilon}\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$

$$-\frac{ig^2}{8\pi^2} T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{1}{12}k^2 g^{\mu\nu} - \tfrac{1}{6}k^\mu k^\nu\right)$$

$$-i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu)\,\delta^{ab}$$

we find:

$$\Pi^{\mu\nu ab}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}$$

gluon self-energy is transverse

$$\Pi(k^2)_{\mathrm{div}} = -(Z_3-1) + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)$$

and so:

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)$$

We found: in Feynman gauge and the $\overline{\mathrm{MS}}$ scheme

not equal!

$$Z_1 = 1 - \Big[C(\mathrm{R}) + T(\mathrm{A})\Big]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)\,,$$

$$Z_2 = 1 - C(\mathrm{R})\,\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)\,,$$

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)\,,$$

Let's calculate the beta function; define:

$$\alpha \equiv \frac{g^2}{4\pi}$$

the dictionary:

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2\tilde{\mu}^\varepsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2\tilde{\mu}^\varepsilon = \frac{Z_{3g}^2}{Z_3^3} g^2\tilde{\mu}^\varepsilon = \frac{Z_{4g}}{Z_3^2} g^2\tilde{\mu}^\varepsilon$$

$$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3}\alpha\tilde{\mu}^\varepsilon$$

Beta functions in quantum electrodynamics

based on S-66

Let's calculate the beta function in QED:

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_1 = Z_1 e\overline{\Psi}A\!\!\!/\Psi + \mathcal{L}_{\mathrm{ct}}\,,$$

$$\mathcal{L}_{\mathrm{ct}} = i(Z_2-1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m-1)m\overline{\Psi}\Psi - \tfrac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu}$$

the dictionary:

$$e_0 = Z_3^{-1/2}Z_2^{-1}Z_1\,\tilde{\mu}^{\varepsilon/2}e$$

$$\alpha = e^2/4\pi$$

$$\alpha_0 = Z_3^{-1}Z_2^{-2}Z_1^2\,\tilde{\mu}^\varepsilon\alpha$$

$$Z_1 = 1 - \frac{\alpha}{2\pi}\,\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\,\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\,\frac{1}{\varepsilon} + O(\alpha^2)$$

Note $Z_1 = Z_2$!

following the usual procedure:

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha$$

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n}$$

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$E_1(\alpha) = \frac{2\alpha}{3\pi} + O(\alpha^2)$$

we find:

$$\beta(\alpha) = \alpha^2 E_1'(\alpha)$$

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3)$$

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3)$$

$$\alpha = e^2/4\pi$$

$$\dot{\alpha} = e\dot{e}/2\pi$$

or equivalently:

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

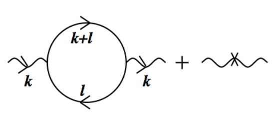
For a theory with N Dirac fields with charges $Q_i e$:

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_{1i}/Z_{2i} = 1$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2) \longrightarrow \textstyle\sum_i Q_i^2 \alpha$$



we find:

$$\beta(e) = \frac{\sum_{i=1}^{N} Q_i^2}{12\pi^2} e^3 + O(e^5)$$

following the usual procedure:

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha$$

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n}$$

$$Z_1 = 1 - \left[C(\mathrm{R}) + T(\mathrm{A})\right] \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\,,$$

$$Z_2 = 1 - C(\mathrm{R}) \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\,,$$

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right] \frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\,,$$

$$G_1(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha}{2\pi} + O(\alpha^2)$$

we find:

$$\beta(\alpha) = \alpha^2 G_1'(\alpha)$$

$$\beta(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\beta(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\alpha = g^2/4\pi$$

$$\dot{\alpha} = g\dot{g}/2\pi$$

or equivalently:

$$\beta(g) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^3}{16\pi^2} + O(g^5)$$

For QCD:

$T(\mathrm{A}) = 3$

$T(\mathrm{R}) = \frac{1}{2}$

$$11 - \tfrac{2}{3}n_{\mathrm{F}}$$

beta function is negative for $n_{\mathrm{F}} \le 16$,
the gauge coupling gets weaker at higher energies!

BRST symmetry

based on S-74

We are going to show that the gauge-fixed lagrangian:

$$\mathcal{L} \equiv \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$$

has a residual form of the gauge symmetry - Becchi-Rouet-Stora-Tyutin symmetry

Consider an infinitesimal transformation for a non-abelian gauge theory:

$$\delta A^a_\mu(x) = -D^{ab}\theta^b(x)$$

$$\delta\phi_i(x) = -ig\theta^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

scalars or spinors in representation R

The BRST transformation is defined as:

$$\delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$

$$= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)$$

$$\delta_{\rm B}\phi_i(x) \equiv igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

we use the ghost field (scalar Grassmann field) instead of $-\theta^a(x)$.

Anything that is gauge invariant is automatically BRST invariant, in particular $\delta_{\rm B}\mathcal{L}_{\rm YM} = 0$!

Now we are going to require that $\delta_{\rm B}\,\delta_{\rm B} = 0$:

this requirement will determine the BRST transformation of the ghost field.

$$\delta_{\rm B}\phi_i(x) \equiv igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^a)(T^a_{\rm R})_{ij}\phi_j - igc^a(T^a_{\rm R})_{ij}\delta_{\rm B}\phi_j$$

-1 for $\delta_{\rm B}$ acting as an anticommuting object

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^a)(T^a_{\rm R})_{ij}\phi_j - g^2c^ac^b(T^a_{\rm R}T^b_{\rm R})_{ik}\phi_k$$

$$c^bc^a = -c^ac^b$$

$$\tfrac{1}{2}[T^a_{\rm R}, T^b_{\rm R}] = \tfrac{i}{2}f^{abc}T^c_{\rm R}$$

Thus we have:

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^c + \tfrac{1}{2}gf^{abc}c^ac^b)(T^c_{\rm R})_{ij}\phi_j$$

that will vanish for all $\phi_j(x)$ if and only if:

$$\delta_{\rm B}c^c(x) = -\tfrac{1}{2}gf^{abc}c^a(x)c^b(x)$$

Now we have to check that $\delta_{\rm B}\,\delta_{\rm B} = 0$ for the gauge field:

$$\delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$

$$= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)$$

$$\delta_{\rm B}(\delta_{\rm B}A^a_\mu) = (\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu)(\delta_{\rm B}c^b) - gf^{abc}(\delta_{\rm B}A^c_\mu)c^b$$

$$= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}(D^{cd}_\mu c^d)c^b$$

$$= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}\underline{(\partial_\mu c^c)}c^b + g^2\underline{f^{abc}f^{cde}}A^e_\mu c^dc^b$$

$$\tfrac{1}{2}(\partial_\mu c^{[c})c^{b]} \equiv \tfrac{1}{2}(\partial_\mu c^c)c^b - \tfrac{1}{2}(\partial_\mu c^b)c^c$$

$$= \tfrac{1}{2}(\partial_\mu c^c)c^b + \tfrac{1}{2}c^c(\partial_\mu c^b)$$

$$= \tfrac{1}{2}\partial_\mu(c^cc^b)\,.$$

$$c^bc^a = -c^ac^b$$

$$= \tfrac{1}{2}(f^{abc}f^{cde} - f^{adc}f^{cbe})$$

$$= -\tfrac{1}{2}[(T^b_{\rm A})^{ac}(T^d_{\rm A})^{ce} - (T^d_{\rm A})^{ac}(T^b_{\rm A})^{ce}]$$

$$= -\tfrac{1}{2}if^{bdh}(T^h_{\rm A})^{ae}$$

$$= -\tfrac{1}{2}f^{bdh}f^{hae}\,,$$

Now we have to check that $\delta_{\rm B}\,\delta_{\rm B} = 0$ for the gauge field:

$$\delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$

$$= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)$$

$$\delta_{\rm B}(\delta_{\rm B}A^a_\mu) = (\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu)(\delta_{\rm B}c^b) - gf^{abc}(\delta_{\rm B}A^c_\mu)c^b$$

$$= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}(D^{cd}_\mu c^d)c^b$$

$$= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}\underbrace{(\partial_\mu c^c)c^b}_{=\frac{1}{2}\partial_\mu(c^cc^b)} + g^2\underbrace{f^{abc}f^{cde}}_{=-\frac{1}{2}f^{bdh}f^{hae}}A^e_\mu c^dc^b$$

$$\delta_{\rm B}(\delta_{\rm B}A^a_\mu) = D^{ab}_\mu(\delta_{\rm B}c^b) - \tfrac{1}{2}gf^{abc}(\partial_\mu c^cc^b) - \tfrac{1}{2}g^2f^{bdh}f^{hae}A^e_\mu c^dc^b$$

$$= D^{ah}_\mu(\delta_{\rm B}c^h) - (\delta^{ah}\partial_\mu - gf^{ahe}A^e_\mu)\tfrac{1}{2}gf^{bch}c^cc^b$$

$$= D^{ah}_\mu(\delta_{\rm B}c^h + \tfrac{1}{2}gf^{bch}c^bc^c)\,.$$

vanishes for the variation of the ghost field we found before:

$$\delta_{\rm B}c^c(x) = -\tfrac{1}{2}gf^{abc}c^a(x)c^b(x)$$

The BRST transformation of the antighost field is defined as:

we treat ghost and antighost fields as independent fields

$$\delta_B \bar{c}^a(x) = B^a(x)$$

B is a scalar field

Lautrup-Nakanishi auxiliary field

then $\delta_B\, \delta_B \;= 0$ implies:

$$\delta_B B^a(x) = 0$$

What is it good for?

We can add to the lagrangian any term that is the BRST variation of some object:

corresponds to fixing a gauge

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \delta_B \mathcal{O}$$

BRST invariant because it is gauge invariant

BRST invariant because $\delta_B(\delta_B\mathcal{O}) \;=\; 0$

Let's choose:

$$\mathcal{O}(x) = \bar{c}^a(x)\left[\tfrac{1}{2}\xi B^a(x) - G^a(x)\right]$$

gauge-fixing function

$$G^a(x) = \partial^\mu A^a_\mu(x)$$

we will get the R_ξ gauge

then

$$\delta_B\mathcal{O} = (\delta_B\bar{c}^a)\left[\tfrac{1}{2}\xi B^a - \partial^\mu A^a_\mu\right] - \bar{c}^a\left[\tfrac{1}{2}\xi(\delta_B B^a) - \partial^\mu(\delta_B A^a_\mu)\right]$$

-1 for δ_B acting as an anticommuting object

$$\begin{aligned}\delta_B A^a_\mu(x) &\equiv D^{ab}_\mu c^b(x)\\ &= \partial_\mu c^a(x) - g f^{abc} A^c_\mu(x) c^b(x)\\ \delta_B \bar{c}^a(x) &= B^a(x)\end{aligned}$$

$$\delta_B\mathcal{O} = \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu + \bar{c}^a\partial^\mu D^{ab}_\mu c^b$$

or

$$\delta_B\mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

$$\delta_B\mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

now we can easily perform the path integral over B:

it is equivalent to solving the classical equation of motion,

$$\frac{\partial(\delta_B\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0$$

and substituting the result back to the formula:

$$\delta_B\mathcal{O} \to -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu\partial^\nu A^a_\nu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

we obtained the gauge fixing lagrangian and the ghost lagrangian

$$S = \int d^4x\, \mathcal{L}$$

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \delta_B\mathcal{O}$$

$$\delta_B\mathcal{O} \to -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu\partial^\nu A^a_\nu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

Symmetries of the complete action:

- Lorentz invariance
- Parity, Time reversal, Charge conjugation
- Global invariance under a given (non-abelian) symmetry group
- BRST invariance
- ghost number conservation (+1 for ghost and -1 for antighost)
- antighost translation invariance $\quad \bar{c}^a(x) \to \bar{c}^a(x) + \chi$

The lagrangian already includes all the terms allowed by these symmetries!

this means that all the divergencies can be absorbed by the Zs of these terms, BRST symmetry requires that the gauge coupling renormalize in the same way in each term.

There is the Noether current associated with the BRST symmetry:

$$j^\mu_{\rm B}(x) = \sum_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_I(x))} \delta_{\rm B} \Phi_I(x)$$

all the fields in the theory

and the corresponding BRST charge:

$$Q_{\rm B} = \int d^3x\, j^0_{\rm B}(x)$$

it is hermitian

the BRST charge generates a BRST transformation:

$$i[Q_{\rm B}, A^a_\mu(x)] = D^{ab}_\mu c^b(x)\,, \qquad \delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$

$$i\{Q_{\rm B}, c^a(x)\} = -\tfrac{1}{2} g f^{abc} c^b(x) c^c(x)\,, \qquad \delta_{\rm B} c^c(x) = -\tfrac{1}{2} g f^{abc} c^a(x) c^b(x)$$

$$i\{Q_{\rm B}, \bar{c}^a(x)\} = B^a(x)\,, \qquad \delta_{\rm B} \bar{c}^a(x) = B^a(x)$$

$$i[Q_{\rm B}, B^a(x)] = 0\,, \qquad \delta_{\rm B} B^a(x) = 0$$

$$i[Q_{\rm B}, \phi_i(x)]_\pm = igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)\,. \qquad \delta_{\rm B}\phi_i(x) \equiv igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

commutator for scalars and anticommutator for spinors

REVIEW

The energy-momentum four-vector is:

$$P^\mu = \int d^3x\, T^{0\mu}(x)$$

Recall, we defined the space-time translation operator

$$T(a) \equiv \exp(-iP^\mu a_\mu)$$

so that

$$T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x-a)$$

we can easily verify it; for an infinitesimal transformation it becomes:

$$[\varphi_a(x), P^\mu] = \tfrac{1}{i}\partial^\mu \varphi_a(x)$$

it is straightforward to verify this by using the canonical commutation relations for $\varphi_a(x)$ and $\Pi_a(x)$.

Since $\delta_{\rm B}\,\delta_{\rm B} = 0$ we have:

$$\boxed{Q^2_{\rm B} = 0}$$

Consider states for which:

$$Q_{\rm B}|\psi\rangle = 0$$

such states are said to be in the Kernel of $Q_{\rm B}$.

Cohomology of $Q_{\rm B}$:

$$|\psi\rangle \neq Q_{\rm B}|\chi\rangle$$

Image of $Q_{\rm B}$:

$$|\psi\rangle = Q_{\rm B}|\chi\rangle$$

we identify two states as a single element of the cohomology if their difference is in the image:

$$|\psi'\rangle = |\psi\rangle + Q_{\rm B}|\zeta\rangle$$

zero norm states:

$$\langle\psi|\psi\rangle = \langle\psi|Q_{\rm B}|\chi\rangle = 0$$

Q is hermitian: $\langle\psi|Q_{\rm B} = 0$.

Consider a normalized state in the cohomology:

$$\langle\psi|\psi\rangle = 1,\; Q_{\rm B}|\psi\rangle = 0,\; |\psi\rangle \neq Q_{\rm B}|\chi\rangle$$

since the lagrangian is BRST invariant:

$$[H, Q_{\rm B}] = 0$$

and so the time evolved state is still annihilated by $Q_{\rm B}$:

$$Q_{\rm B} e^{-iHt}|\psi\rangle = e^{-iHt} Q_{\rm B}|\psi\rangle = 0$$

(in addition, a unitary time evolution does not change the norm of a state)

the time-evolved stay must still be in the cohomology!

We will see shortly that the physical states of the theory correspond to the cohomology of $Q_{\rm B}$!

All states in the theory can be generated from creation operators (we start with widely separated wave packets, and so we can neglect interaction):

$$A^\mu(x) = \sum_{\substack{\lambda=>,<,\\+,-}} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu *}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \varepsilon_\lambda^\mu(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) e^{-ikx}\right]$$

we are neglecting group indices

$$c(x) = \int \widetilde{dk} \left[c(\mathbf{k}) e^{ikx} + c^\dagger(\mathbf{k}) e^{-ikx}\right],$$

$$\bar{c}(x) = \int \widetilde{dk} \left[b(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}\right],$$

$$\phi(x) = \int \widetilde{dk} \left[a_\phi(\mathbf{k}) e^{ikx} + a_\phi^\dagger(\mathbf{k}) e^{-ikx}\right],$$

represents matter fields

for $k^\mu = (\omega, \mathbf{k}) = \omega(1,0,0,1)$ four polarization vectors can be chosen as:

$$\varepsilon_>^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,1),$$

$$\varepsilon_<^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,-1),$$

$$\varepsilon_+^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,-i,0),$$

$$\varepsilon_-^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,+i,0).$$

Setting $g=0$ and matching coefficients of e^{-ikx} we find:

$$i[Q_\mathrm{B}, A_\mu^a(x)] = D_\mu^{ab} c^b(x),$$

$$i\{Q_\mathrm{B}, c^a(x)\} = -\tfrac{1}{2} g f^{abc} c^b(x) c^c(x),$$

$$i\{Q_\mathrm{B}, \bar{c}^a(x)\} = B^a(x),$$

$$i[Q_\mathrm{B}, B^a(x)] = 0,$$

$$i[Q_\mathrm{B}, \phi_i(x)]_\pm = igc^a(x)(T_\mathrm{R}^a)_{ij}\phi_j(x).$$

$$A^\mu(x) = \sum_{\substack{\lambda=>,<,\\+,-}} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu *}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \varepsilon_\lambda^\mu(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) e^{-ikx}\right]$$

$$c(x) = \int \widetilde{dk} \left[c(\mathbf{k}) e^{ikx} + c^\dagger(\mathbf{k}) e^{-ikx}\right],$$

$$\bar{c}(x) = \int \widetilde{dk} \left[b(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}\right],$$

$$\phi(x) = \int \widetilde{dk} \left[a_\phi(\mathbf{k}) e^{ikx} + a_\phi^\dagger(\mathbf{k}) e^{-ikx}\right],$$

$$\varepsilon_>^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,1),$$

$$\varepsilon_<^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,-1),$$

$$\varepsilon_+^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,-i,0),$$

$$\varepsilon_-^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,+i,0).$$

$$[Q_\mathrm{B}, a_\lambda^\dagger(\mathbf{k})] = \sqrt{2}\omega\, \delta_{\lambda>}\, c^\dagger(\mathbf{k}),$$

$$\{Q_\mathrm{B}, c^\dagger(\mathbf{k})\} = 0,$$

$$\{Q_\mathrm{B}, b^\dagger(\mathbf{k})\} = \xi^{-1}\sqrt{2}\omega\, a_<^\dagger(\mathbf{k}),$$

$$[Q_\mathrm{B}, a_\phi^\dagger(\mathbf{k})] = 0.$$

we also use EM to eliminate B:

$$\frac{\partial(\delta_\mathrm{B}\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A_\mu^a(x) = 0$$

Consider a normalized state in the cohomology: $\langle\psi|\psi\rangle = 1,\ Q_\mathrm{B}|\psi\rangle = 0,$

the state $c^\dagger(\mathbf{k})|\psi\rangle$ is proportional to $Q_\mathrm{B} a_>^\dagger(\mathbf{k})|\psi\rangle$ and so it is not in the cohomology

if we add a photon with polarization >, the state $a_>^\dagger(\mathbf{k})|\psi\rangle$ is not annihilated by Q and so it is not in the cohomology

$$[Q_\mathrm{B}, a_\lambda^\dagger(\mathbf{k})] = \sqrt{2}\omega\, \delta_{\lambda>}\, c^\dagger(\mathbf{k}),$$

$$\{Q_\mathrm{B}, c^\dagger(\mathbf{k})\} = 0,$$

$$\{Q_\mathrm{B}, b^\dagger(\mathbf{k})\} = \xi^{-1}\sqrt{2}\omega\, a_<^\dagger(\mathbf{k}),$$

$$[Q_\mathrm{B}, a_\phi^\dagger(\mathbf{k})] = 0.$$

the state $b^\dagger(\mathbf{k})|\psi\rangle$ is not annihilated by Q and so it is not in the cohomology

the state $a_<^\dagger(\mathbf{k})|\psi\rangle$ is proportional to $Q_\mathrm{B} b^\dagger(\mathbf{k})|\psi\rangle$ and so it is not in the cohomology

states: $a_+^\dagger(\mathbf{k})|\psi\rangle$, $a_-^\dagger(\mathbf{k})|\psi\rangle$ and $a_\phi^\dagger(\mathbf{k})|\psi\rangle$ are annihilated by Q but cannot be written as Q acting on some state and so they are in the cohomology!

the vacuum is also in the cohomology

Thus we found:

we can build an initial state of widely separated particles that is in the cohomology only with matter particles and photons with polarizations + and -. No ghosts or >,< polarized photons can be produced in the scattering process (a state in the cohomology will evolve to another state in the cohomology).