Lectures on N = 2 String Theory

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ABSTRACT

Starting from an arbitrary $N = 2$ superconformal field theory it is described how a fully consistent, space–time supersymmetric heterotic–like string theory in an even number of dimensions is constructed. Four dimensional theories which arise in this construction have a gauge group which contains $E_8 \times E_6$ with chiral fermions in the 27 and $\bar{27}$ representations of E_6, and thus are phenomenologically viable. The explicit massless spectrum is studied for particular solvable examples. It is shown that such spectra are the typical ones expected from the field theory compactification on manifolds of vanishing first Chern class. It is concluded that all 'N=2 string theories' describe string propagation on such manifolds. An explicit calculation of the Yukawa couplings 27^3 is described for all the string theories in which the $N = 2$ superconformal theory affords a scalar description. The result of this calculation is shown to be geometric.

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.1. Introduction

The study of four dimensional string theory is central to the idea that strings might provide a framework for unification. Initially, it was hoped that the internal consistency of first quantized string theory would be sufficiently restrictive to pinpoint the correct theory, much as it is for ten dimensional string theory. However, it was soon realized that this is not the case, and at least, for closed bosonic strings, any internal conformal field theory can be used in compactification.

A number of compactification schemes based on free field theory were proposed. The simplest one is the toroidal compactification [1]. More realistic theories are obtained by projections of toroidal theories (orbifolds) [2]. Models based on free fermions were studied in ref. [3].

There are a number of reasons for studying string theories constructed from non–free conformal field theories. First, the space of conformal field theories in two dimensions is huge, and the free field theories are only a small subset of it. Thus string theories based on interacting conformal field theory offer a much richer set of possibilities to explore for realistic model building. Second, the general compactification on such conformal field theories can offer a unified understanding that would otherwise be quite obscure in disconnected examples. Last, the study of non-trivial string theory may enhance the understanding of conformal field theory itself, an important goal by its own right.

As was demonstrated, for example, by the study of closed bosonic strings propagating on group manifolds [4], a consistent closed bosonic string theory is more or less guaranteed to arise by the structure of conformal field theory. The most important restriction on the possibilities comes from the modular invariance of the one–loop amplitude. Modular invariance is obeyed for bosonic compactifications if one takes the full spectrum of the conformal field theory for the construction of the string.

For phenomenological reasons one would actually like to study heterotic–like string theories in four dimensions, in which the left moving Minkowski degrees of freedom are fermionic strings (with total central charge $c = 12$ in the light–cone gauge) and the right moving are bosonic ($c = 24$). Another requirement one might wish to impose on the theory is that of space–time supersymmetry in four dimensions. Space-time supersymmetry can solve various phenomenological questions (such as the hierarchy problem) in addition to its theoretical advantages like the absence of tachyons.

Both the heterosis and the space–time supersymmetry seem to be in apparent contradiction with the requirement of modular invariance in a general conformal field theory. The only generic way to achieve modular invariance in a non–free conformal field theory is to have a left–right symmetric spectrum. However, the heterotic–like string is asymmetric from its very nature, where even the trace anomalies of the left movers and the right movers are different. Also, space–time supersymmetry requires the appearance of superpartners in the spectrum, spoiling the left–right symmetry.

There is a more or less unique solution to both issues. For any $N = 2$ superconformal field theory with $c = 9$ one can construct a fully consistent closed superstring theory with space–time supersymmetry [5]. The space–time supersymmetry generator is given by $Q = \exp(i\phi)$ where ϕ is the $U(1)$ boson of the $N = 2$ algebra. Such a generator was first seen in the leading $1/R$ term (where R is the radius) of Calabi–Yau manifolds [6]. In ref. [7] it was noted that the internal part of this field may be represented as a twist of the $N = 2$ algebra ('spectral flow'). The implementation of supersymmetry follows from a new supersymmetry projection [5] in which one discards all states for which the total $U(1)$ charge is not an odd integer. If one imposes at the same time supersymmetry in the spectrum, modular invariance is restored. We review the construction of this projection in section (2).

The solution to the problem of heterosis is provided by a map that takes any consistent superstring–like theory in any even dimension to a consistent heterotic–like string theory with the gauge groups $E_8 \times SO(8 + d)$ or $SO(24 + d)$ where

d is the number of transverse dimensions [5]. In the supersymmetric case and $d = 2$ (four dimensions) the gauge group becomes $E_8 \times E_6$ or $SO(26)$. This map is discussed in section (3).

The resulting supersymmetric heterotic–like string theories, which will be referred to as $N = 2$ string theories, have chiral fermions in the 27 and $\bar{27}$ of E_6 and are thus phenomenologically viable. To construct explicit examples of such theories we turn to solvable conformal field theory in two dimensions. A relatively simple family of such theories is provided by the minimal realizations of the $N = 2$ superconformal algebra [8, 7, 9], for which the trace anomaly is

$$c = \frac{3k}{k+2}, \qquad \text{for } k = 1, 2, 3 \ldots . \tag{1}$$

The minimal theories are discussed in sections (4) and (5).

Using the minimal theories as building blocks, the supersymmetry projection and the map to heterosis, gives a consistent solvable string theory for any combination of the minimal models with the correct central charge. The partition function of these minimal $N = 2$ string theories are considered in section (6).

The massless spectra of such theories are studied in section (7). Using discrete symmetries as a tool in classifying states, it is shown that the resulting spectra agree exactly with manifolds of vanishing first Chern class. Compactifications of the field theory limit of the heterotic string on such manifolds were initiated in a beautiful work by Candelas et al. [10] and were subsequently studied extensively [11]. By making a comparison with the results of Candelas et al. it is established that all $N = 2$ string theories describe string propagation on manifolds of vanishing first Chern class. This section is based on ref. [12].

In section (8) we discuss scalar field theory formulations of $N = 2$ superconformal field theories. It is shown that the structure constants for the chiral fields in the theory can be computed exactly for any such theory. Using this result we compute the Yukawa couplings of the type 27^3. It is shown that these couplings agree exactly with a field theory geometrical formula [13].

.2. Space–time supersymmetry and superconformal invariance

Consider the most general superstring compactification to $D = d + 2$ dimensions. We assume that D is even, d is the number of transverse dimensions. The total central charge of the theory in the light cone gauge is $c = 12$. The space time theory is composed of d free bosons and d free fermions on the world sheet. Since each free boson contributes $c = 1$ to the trace anomaly and each fermion contributes $1/2$, the trace anomaly of the space–time degrees of freedom is $c = d + d/2 = 3d/2$. The trace anomaly of the internal theory is thus $12 - 3d/2$. In particular, in order to compactify to four dimensions we need an internal theory with the trace anomaly $c = 9$.

The consistency of the superstring theory requires that the internal theory is not only conformally invariant, but also has an $N = 1$ superconformal invariance in two dimensions. In what follows, we shall usually assume that the internal theory actually has an $N = 2$ superconformal symmetry. The reason is that this will enable us to achieve space–time supersymmetry.

The $N = 2$ superconformal algebra contains in addition to the usual moments of the stress tensor $T(z) = \sum L_n z^{-n-2}$, two fermionic superpartners, $G^{\pm}_n$ and a $U(1)$ current whose moments we denote by J_n. $G^{\pm}(z) = G^{\pm}_n z^{-n-3/2}$ and $J(z) = J_n z^{-n-1}$. The values of the indices n depend on the boundary condition assumed for the superstress tensors,

$$G^{\pm}(e^{2\pi i} z) = e^{\pm 2\pi i \eta} G^{\pm}(z), \tag{2}$$

where $\eta = 0$ corresponds to the Neveu–Schwarz sector and $\eta = \frac{1}{2}$ to the Ramond sector. Accordingly, the indices n in $G^{\pm}_n$ take values in $Z + \frac{1}{2} \pm \eta$, where Z denotes the integers.

The commutation relations of the $N = 2$ algebra are given by,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^2 - m)\delta_{n+m,0}, \\
[L_m, J_n] &= -nJ_{m+n}, \\
[L_m, G_r^{\pm}] &= (\frac{m}{2} - r)G_{m+r}^{\pm}, \\
[J_m, J_n] &= -nJ_{n+m}, \\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}.
\end{aligned} \tag{3}$$

The first equation is the usual Virasoro algebra for the moments of the stress–tensor. The second and third commutation relations simply imply that $G^{\pm}(z)$ and $J(z)$ are primary fields of the Virasoro algebra with the dimensions $\frac{3}{2}$ and 1, respectively. The fourth relation implies that $J(z)$ is a free boson $U(1)$ current, $J = i\sqrt{\frac{c}{3}}\partial_z \phi$ where ϕ is a canonical free boson; the J_n's are the usual bosonic creation and annihilation operators. The fifth equation implies that $G^{\pm}(z)$ have the $U(1)$ charges ± 1. The only new commutation relation is the last one involving the superstress tensors. All the indices take value in the integers, except r and s which take their values in $Z + \frac{1}{2} \pm \eta$, as explained above.

For every value of η we get a different algebra, $0 \leq \eta < 1$. Algebraically, although not physically, the algebras obtained for different values of η are isomorphic to each other [14]. To see this one makes the change of variables,

$$\begin{aligned}
L_n' &= L_n + \eta J_n + \frac{1}{6}\eta^2 c\delta_{n,0}, \\
J_n' &= J_n + \frac{1}{3}\eta c\delta_{n,0}, \\
(G_r^{\pm})' &= G_{r\pm\eta}^{\pm}.
\end{aligned} \tag{4}$$

It is easy to check that the primed variables obey the same commutation relations as the unprimed ones, with a shift of the moding by η.

The isomorphism of the algebras for different boundary conditions has a natural interpretation. Consider the $U(1)$ current algebra generated by $J(z)$. As noted earlier J may be bosonized,

$$J = i\sqrt{\frac{c}{3}}\partial_z\phi, \tag{5}$$

where ϕ is a canonical free boson. Now, an arbitrary field in the theory, f, with the $U(1)$ charge q may be written as

$$f = \hat{f}e^{iq\sqrt{\frac{3}{c}}\phi}, \tag{6}$$

where the field $\hat{f}$ is neutral. It follows from eq. (6) that all the fields in a unitary superconformal field theory obey

$$\Delta \geq \frac{3q^2}{2c}, \tag{7}$$

where the equality holds for the fields for which $\hat{f} = 1$. If f is a primary field of the $U(1)$ current algebra,

$$J(z)f(w) = \frac{qf(w)}{z-w} + \text{regular terms}, \tag{8}$$

then the field $\hat{f}$ commutes with the $U(1)$ current algebra. In other words, every $N = 2$ superconformal field theory, with the central charge c, is a product of a $U(1)$ current algebra with $c = 1$ and some quotient theory whose central charge is $c - 1$. In particular, the fields $G^{\pm}(z)$ may be written as,

$$G^{\pm}(z) = \hat{G}^{\pm}(z)e^{\pm i\sqrt{3/c}\phi}, \tag{9}$$

where $\hat{G}^{\pm}$ is a field in the $c - 1$ conformal field theory.

Now, for every field, f, in the Neveu–Schwarz sector ($\eta = 0$) we can write a field in the sector twisted by η,

$$f_\eta(z) = \hat{f}(z)e^{i\phi(q\sqrt{\frac{3}{c}}+\eta\sqrt{c/3})}. \tag{10}$$

It is straightforward to check that the O.P.E. of the field $f_\eta(z)$ with $G^{\pm}(w)$ contains the terms $(z-w)^{n\pm\eta}$, where the n are integers, implying that the field

f_η is in the sector twisted by η. The dimension and $U(1)$ charge of the field f_η can be read from those of the field f. We find,

$$q' = q + \eta c/3, \tag{11}$$

$$\Delta' = \Delta + \frac{1}{2}(q\sqrt{3/c} + \eta\sqrt{c/3})^2 - \frac{3q^2}{2c} = \Delta + \eta q + \frac{1}{6}\eta^2 c. \tag{12}$$

Comparing eqs. (11–12) with eq. (4) we see that these two expressions are identical. We conclude that the isomorphism eq. (4) corresponds to nothing but a product with a free boson exponential.

Consider now some $N = 2$ superconformal field theory with the central charge c. The Hilbert space of the theory, $\mathcal{H}$, decomposes into a set of representations of the left and right $N = 2$ algebras. The number of representations to which $\mathcal{H}$ decomposes may be finite or infinite,

$$\mathcal{H} = \oplus_{p,q}\mathcal{H}_{p,q}, \tag{13}$$

where p and q label the left and right representations. It is actually convenient to split every representation of the $N = 2$ algebra into two representations of the subalgebra generated by an even number of $G^\pm$. The reason for this will become clear in the sequel. Since acting with an even number of G changes the dimension of a field by an integer and its $U(1)$ charge by an odd integer, we can assume without loss of generality that all the dimensions of the fields in the representation $\mathcal{H}_p$ differ from one another by an integer, and, similarly, the charges differ by an even integer. We shall denote by $\Delta_p \bmod 1$ and $Q_p \bmod 2$ the common dimension and charge of the fields in the representation $\mathcal{H}_p$.

For each representation, $\mathcal{H}_p$, we may define the character as the function

$$\chi_p(\tau, z, u) = e^{-2\pi i u} \operatorname*{Tr}_{\mathcal{H}_p} e^{2\pi i z J_0} e^{2\pi i \tau (L_0 - \frac{c}{24})}, \tag{14}$$

which is a generating function for the number of states in the representation with a given dimension and $U(1)$ charge,

$$\chi_p(\tau, z, u) = e^{-2\pi i u} \sum_{q,\Delta} \text{mult}(q, \Delta) e^{2\pi i z q + 2\pi i \tau (\Delta - c/24)}, \tag{15}$$

where $\text{mult}(q, \Delta)$ is the number of states in the representation with the charge q and dimension Δ.

The 1–loop partition function of the theory is

$$Z = \sum_{p,q} N_{p,q} \chi_p(\tau, 0, 0) \chi_q(\tau, 0, 0)^*, \tag{16}$$

where $N_{p,q}$ denotes the number of times the representation $\mathcal{H}_{p,q}$ appears in the spectrum. Under modular transformations, $\tau \to \frac{a\tau+b}{m\tau+n}$, the partition function $Z(\tau)$ must stay invariant. In particular, the typical situation is that the characters $\chi_p(\tau, 0, 0)$ form a unitary representation of the modular group,

$$\chi_p \left(\frac{a\tau + b}{m\tau + n}, 0, 0 \right) = M \begin{pmatrix} a & b \\ m & n \end{pmatrix}_{p,q} \chi_q(\tau, 0, 0), \tag{17}$$

where the matrix M is unitary.

Define an action of the modular group on the variables (τ, z, u) by,

$$(\tau, z, u)|_M = \left(\frac{a\tau + b}{m\tau + n}, \frac{z}{m\tau + n}, u + \frac{cz^2}{6(m\tau + n)} \right), \qquad \text{where } M = \begin{pmatrix} a & b \\ m & n \end{pmatrix}, \tag{18}$$

and c is the central charge of the theory. An important point is that the full character $\chi_p(\tau, z, u)$ transforms under the action of the modular group defined in eq. (18) in precisely the same way as the specialized characters, eq. (17),

$$\chi_p(\frac{a\tau + b}{m\tau + n}, \frac{z}{m\tau + n}, u + \frac{cz^2}{6(m\tau + n)}) = S \begin{pmatrix} a & b \\ m & n \end{pmatrix}_{p,q} \chi_q(\tau, z, u). \tag{19}$$

In the sequel, when discussing examples of $N = 2$ superconformal field theories we shall see that indeed eq. (19) is satisfied. We will give below a justification for the transformation law eq. (19) in the general $N = 2$ superconformal field theory.

Consider a general $N = 2$ superconformal field theory with some central charge c in the Neveu–Schwarz ($\eta = 0$) sector. Let $\mathcal{H}_p$ denote some representation. As discussed earlier, from each field in this representation, f, we can get a new one by multiplying it with an exponential of the $U(1)$ free boson, eq. (10). This new state is in the representation of the η-twisted $N = 2$ superconformal algebra. The full character of the resulting η twisted representation, $\mathcal{H}_p^\eta$, is,

$$\chi_p^\eta(\tau, z, u) = \operatorname*{Tr}_{\mathcal{H}_p^\eta} e^{-2\pi i u} e^{2\pi i z J_0'} e^{2\pi i \tau (L_0' - c/24)} = \chi_p(\tau, z + \eta\tau, u - \frac{1}{6}\eta^2 \tau c - \frac{1}{3}\eta z c), \tag{20}$$

where we used eqs. (11-12).

Consider now the behavior under the modular transformation $\tau \to -1/\tau$ of $\chi_p^\eta(\tau, 0, 0)$. From the transformation law, eq. (19), we find,

$$\chi_p(-\frac{1}{\tau}, 0, 0) = S_{pq} \chi_q(\tau, \eta, 0), \qquad \text{where } S_{pq} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{pq}, \tag{21}$$

which can be written as

$$\chi_p(\tau, \eta, 0) = \operatorname*{Tr}_{\mathcal{H}_p} e^{2\pi i \eta J_0} e^{2\pi i \tau (L_0 - \frac{c}{24})}. \tag{22}$$

The partition function, eq. (22), has a very natural physical interpretation. It is the torus partition function in the representation $\mathcal{H}_p$ with a charge operator inserted in the time direction. The charge is defined as η times the $U(1)$ charge. Thus, this is the partition function for the sector twisted by η in the time direction. On the other hand, $\chi_p^\eta(\tau, 0, 0)$ is the partition function for the theory twisted by η in the space direction. That the two partition functions get exchanged under the modular transformation $\tau \to -\frac{1}{\tau}$ is indeed precisely the correct behavior. Thus we see that the transformation law, eq. (19), is compatible with the modular transformations of the torus partition functions in the twisted sectors of the theory.

Assume now that the conformal field theory under discussion has the central charge $c = 12$, and that it is a tensor product of a d dimensional superstring with a $12 - 3d/2$ internal $N = 2$ superconformal field theory. The superstring degrees of freedom describe a conformal field theory with $c = 3d/2$ composed out of d transverse bosons, X^μ, and d transverse fermions, ψ^μ. When d is even, this conformal field theory has an $N = 2$ superconformal invariance. Grouping X^μ and ψ^μ into $d/2$ complex fermions and bosons, the $N = 2$ currents can be written as, $J = \psi_i^* \psi_i$, $G^+ = -\sqrt{2}\psi_i \partial_z X_i$ and $G^- = -\sqrt{2}\psi_i^* X_i^*$, where $i = 1, 2, \ldots, \frac{d}{2}$. Thus the total $c = 12$ theory also has an $N = 2$ superconformal invariance.

The field

$$Q = e^{i\frac{\phi}{2}\sqrt{\frac{c}{3}}}, \tag{23}$$

where ϕ is the bosonized total $U(1)$ current of the total ($c = 12$) theory, is a space–time fermion, as we shall see in the sequel. Consider the action of the field

$$Q^n = e^{i\frac{1}{2}\sqrt{\frac{c}{3}}n\phi} \tag{24}$$

on the character $\chi_p(\tau, z, u)$. From eq. (20) we find,

$$Q^n \chi(\tau, z, u) = \chi_p(\tau, z + \frac{n\tau}{2}, u - \frac{n^2 c\tau}{24} - \frac{znc}{6}). \tag{25}$$

From the partition function χ_p we can form a 'supersymmetrized' partition function by summing over all states related by the action of the supersymmetry charge, Q. Explicitly, the supersymmetrized partition function is,

$$\chi_p^{\rm sum} = \sum_{n\in Z} (-1)^n Q^n \chi_p(\tau, z, u) = \sum_{n\in Z} (-1)^n \chi_p(\tau, z + \frac{n\tau}{2}, u - \frac{n^2 c\tau}{24} - \frac{znc}{6}). \tag{26}$$

Now, consider the action of the modular transformation $\tau \to -\frac{1}{\tau}$ on the supersymmetrized partition function $\chi_p^{\rm sum}$. From the transformation law, eq.

(19), we find,

$$\chi_p^{\rm sum}(-\frac{1}{\tau},0,0) = S_{pq}\sum_{n\in Z}(-1)^n\chi_q(\tau,\frac{n}{2},0). \tag{27}$$

The sum on the r.h.s of eq. (27) has a very simple interpretation. Using the fact that all the states in the representation $\mathcal{H}_q$ have the same $U(1)$ charge up to an even integer, $Q_q \bmod 2$, we find,

$$\sum_{n\in Z}(-1)^q\chi_q(\tau,\frac{n}{2},0) = \mathop{\rm Tr}_{\mathcal{H}_p} e^{2\pi i(\tau-c/24)}\sum_{n\in Z}(-1)^n e^{\pi i n J_0} = S_{pq}\delta(Q_q)\chi_q(\tau,0,0). \tag{28}$$

where we denote by $\delta(Q_q)$ a delta function which vanishes unless the $U(1)$ charge, Q_q, is an odd integer, in which case it is equal to one. In other words, the supersymmetrized partition function, $\chi_p^{\rm sum}$ transforms under $\tau\to-\frac{1}{\tau}$ into exactly the same sum of partition functions as the original partition function, missing only all the representations with a total $U(1)$ charge which is not an odd integer.

Eq. (28) shows also that the supersymmetrized partition function $\chi_p^{\rm sum}$ is actually a certain sum of characters of the untwisted theory, since it is a modular transform of such a sum. Thus it is a bona fide partition function of a conformal field theory. In addition, the minus signs in the definition eq. (26) are precisely the correct ones to guarantee spin–statistics. The field Q takes a space–time boson to a space–time fermion and vice versa. The space time fermions must have a negative sign in the partition function, and indeed from eq. (26) we see that multiplying by Q precisely flips the sign in the partition function.

As we saw above, the supersymmetrization and the condition of having only odd integral $U(1)$ charges are dual under the modular transformation $\tau\to-1/\tau$. Thus, we can get a partition function which is invariant under $\tau\to-\frac{1}{\tau}$ by imposing both conditions at the same time. Precisely for $c=12$ the field Q changes the total $U(1)$ charge by an even integer and thus it is consistent to do so. Consider then the partition functions, $\delta(Q_p)\chi_p^{\rm sum}$. Under the S modular transformations these partition functions transform with the same unitary matrix as before, S_{pq}. Thus we may form a fully modular invariant partition function by imitating the original partition function, eq. (16),

$$Z_{\rm susy} = \sum_{p,q} N_{pq} \chi_p^{\rm sum} \chi_q^{\rm sum *}, \tag{29}$$

where the sum extends over all the left and right representations with the $U(1)$ charge which is an odd integer.

Finally, to insure that the full partition function $Z_{\rm susy}$ is modular invariant we need to check only the invariance under the generator $\tau \to \tau + 1$. From, eq. (14), we find that $\chi_p^{\rm sum}$ transforms under $\tau \to \tau + 1$ as,

$$\begin{aligned} &\chi_p^{\rm sum}(\tau+1,0,0) = \\ &\sum_{n\in Z} (-1)^n e^{2\pi i(\Delta_p - \frac{c}{24})} e^{\pi i n Q_p} e^{-2\pi i \frac{n^2 c}{24}} \chi_p^{\rm sum}(\tau,0,0) = e^{2\pi i(\Delta_p - \frac{c}{24})} \chi_p^{\rm sum}(\tau,0,0), \end{aligned} \tag{30}$$

where we have used, crucially, the fact that $c = 12$. So we see that precisely for $c = 12$ (actually, $c = 12 \bmod 24$) the partition functions $\chi_p^{\rm sum}$ transform under $\tau \to \tau + 1$ in precisely the same way as the original partition functions. We thus conclude that the supersymmetrized partition function $Z_{\rm susy}$ is fully modular invariant and describes a physical, d dimensional superstring theory.

We will now prove that the string theory described by the partition function, eq. (29), is indeed space–time supersymmetric. In particular, at each mass level of the string there should be an equal number of bosonic and fermionic excitations.

Consider the field Q, eq. (23). The total $N = 2$ algebra is composed of the internal $N = 2$ algebra, whose $U(1)$ current we denote by J_i and the space–time $U(1)$ current, $J_{\rm s.t.} = \psi^*_\mu \psi_\mu$. For a compactification to four dimension, $c_i = 9$ and $c_{\rm s.t.} = 3$. We can also bosonize the space–time and internal $U(1)$ currents,

$J_i = i\sqrt{3}\partial_z\phi_i$ and $J_{\rm s.t.} = i\partial_z\phi_{\rm s.t.}$. From $J = J_i + J_{\rm s.t.}$ it follows that,

$$\phi = \frac{\sqrt{3}\phi_i + \phi_{\rm s.t.}}{2}. \tag{31}$$

Substituting into eq. (23) we realize that Q may be written as,

$$Q = e^{\frac{i\phi_{\rm s.t.}}{2}} e^{\frac{i\sqrt{3}\phi}{2}} = S e^{\frac{i\sqrt{3}\phi}{2}}, \tag{32}$$

where S is the spin field of the $SO(2)$ current algebra. Similarly, the complex conjugate field can be written as,

$$Q^\dagger = S^\dagger e^{\frac{-i\phi\sqrt{3}}{2}}. \tag{33}$$

Similar formulas hold in dimensions greater than four, where S stands for the highest weight component of the spin field.

Now, the field $Q = e^{i\phi}$ may be fermionized. Up to irrelevant cocycle factors Q is a free Dirac fermion on the world sheet. The fermion number is equal to half the $U(1)$ charge. Since all the fields in the theory have odd integral $U(1)$ charges, it follows that all the fields are in the Ramond sector of this free fermion theory. (This R sector should not be confused with the R and NS sectors of the $N = 2$ algebra.) The spectrum of the theory is invariant under the action of Q and $Q^\dagger$ (defined in the operator product sense; equivalently we may define this action on the states via the moments of Q and $Q^\dagger$ which form the usual Ramond algebra). This follows from the expression of the partition function, eq. (26), and the invariance under the $U(1)$ current $J = 2i\partial_z\phi = :Q^\dagger Q:$. Now, the $c = 12$ conformal field theory under discussion decomposes into this free fermion theory times a $c = 11$ conformal field theory. Since all the fields in the R sector of a free fermion theory can be obtained by acting with Q and $Q^\dagger$ on the unique highest weight state, the theory is a tensor product of these two sub–theories, $Z(\tau) = Z_f(\tau)Z_i(\tau)$, where $Z(\tau)$ is the total partition function, $Z_f(\tau)$ is the fermionic partition function and $Z_i(\tau)$ is the partition function of the $c = 11$ theory. The free fermion theory is invariant under an $SO(2)$ current algebra. There are two partition functions for this algebra in the Ramond sector, the spinor (helicity $\frac{1}{2}$) and the anti–spinor (helicity $-\frac{1}{2}$). Explicitly, these partition functions are

$$\frac{\Theta_{\pm 1,2}(\tau,0,0)}{\eta(\tau)}, \tag{34}$$

where the Θ functions are the classical $SU(2)$ theta functions at level 2 (we will discuss these further in section (5)).

Now, we can get from the spinor representation to the anti–spinor by the action of an odd number of Q or $Q^\dagger$. So the spectrum must contain both representations. However, in acting with Q or $Q^\dagger$ the partition function changes sign. Thus the partition function of the free fermion part is,

$$Z_f = \frac{\Theta_{1,2}(\tau,0,0) - \Theta_{-1,2}(\tau,0,0)}{\eta(\tau)} = 0. \tag{35}$$

It follows that the total partition function $Z = Z_f Z_i$ vanishes as well. Since in the partition function the space–time bosons come with a plus sign and the space–time fermions with a minus sign, the vanishing of the partition function implies that at each mass level there is an equal number of space–time fermions and space–time bosons. The proof is the same for any even space–time dimension⋆.

The full supersymmetry algebra follows from the additional invariance under the $SO(d)$ current algebra. Thus we have the invariance with respect to the fields,

$$Q^\alpha = S_\alpha e^{i\phi_i\sqrt{3}/2}, \qquad Q^{\alpha\dagger} = S^\dagger_\alpha e^{-i\phi_i\sqrt{3}/2}. \tag{36}$$

where S_α is the full spin field and $S^\dagger_\alpha$ is its complex conjugate. The full super-

⋆ Since, the general supersymmetry projection depends only on the total $c = 12$ $N = 2$ superconformal algebra, one can use it also in odd number of space–time dimensions, provided the total theory has $N = 2$ superconformal invariance. Since one needs to complete the space–time theory to an $N = 2$ theory, this is possible only by compactifying an odd number of dimensions on a torus.

symmetry generators in the light–cone gauge are proportional to these fields and are given, in four dimensions, by the zero modes of the fields

$$
\begin{aligned}
S^{\alpha} &= \frac{1}{\sqrt{p^+}}(\partial_z X^1 + i\partial_z X^2)Q^{\alpha}, \\
S^{\dot{\alpha}} &= \frac{1}{\sqrt{p^+}}(\partial_z X^1 - i\partial_z X^2)Q^{\alpha\dagger},
\end{aligned}
\tag{37}
$$

where X^1 and X^2 are the two free transverse bosons and p^+ is the light–cone momenta. The commutation relations of these supersymmetry generators follow from the OPE of the fields Q and $Q^\dagger$ which are essentially the same in any even dimension since they depend only on the free boson of the total $N = 2$ algebra. For a recent discussion of this see [15]. One might wonder whether supersymmetry can be obtained in a different way than the one described above. Interestingly, in ref. [16] a no–go theorem was discussed showing that supersymmetry requires the $N = 2$ superconformal invariance.

Actually, in the superstring partition function, eq. (29), there is also another supersymmetry coming from the right movers. Thus the superstring theory has an $N = 2$ space–time supersymmetry. If, as in the sequel, we will choose the right movers to be bosonic, then there will only be an $N = 1$ supersymmetry.

Consider now the massless spectrum of the string theory in four dimensions. These states have the left dimension $\Delta = \frac{1}{2}$. Using eq. (7) we find that $|q| \leq 2$ for these states. Since q is an odd integer it follows that $q = +1$ or $q = -1$. There are four possible representations for the $SO(2)$ current algebra. These are the singlet, the vector, the spinor and the anti–spinor representations (helicities 0, 1 and $\pm\frac{1}{2}$). The dimensions and $U(1)$ charges for each of these representations are listed in table 1. In this table Δ_c and Q_c stand for the dimension and $U(1)$ charge of the space–time part. Δ_i and Q_i stand for the dimension and $U(1)$ charge of the field from the internal conformal field theory needed in order to complete the total dimension to $\frac{1}{2}$ and the total $U(1)$ charge to ± 1.

Table 1.

Dimensions and charges for the left movers.

Representation	Δ_c	Q_c	Δ_i	Q_i
Singlet	0	0	$\frac{1}{2}$	± 1
Vector	$\frac{1}{2}$	± 1	0	0
Spinor	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{2}$
Antispinor	$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{3}{8}$	$-\frac{1}{2}$

Let us go over the entries of table 1. Along with the vector of $SO(2)$ we must have a field of dimension zero. There is a unique such field in a unitary conformal field theory which is the identity field. Thus this representation appears exactly once in the massless spectrum. The superpartners of the vector representation are spinor and anti–spinor fields which are Q and $Q^\dagger$. Assuming, as in the sequel, the right movers to be bosonic, these fields would give the graviton and the gravitino, respectively, along with the gauge multiplet.

The singlets of $SO(d)$ are fields with dimension $\frac{1}{2}$ and $U(1)$ charge 1. Denote such a field by C.The field C is primary and obeys $2\Delta = q$. It follows that

$$
\begin{aligned}
&G_-(\zeta)C(z) = O(\frac{1}{\zeta - z}) \\
&G_+(\zeta)C(z) = O(1).
\end{aligned}
\tag{38}
$$

Fields for which eq. (38) holds are called chiral primary fields. Similarly, the fields for which $2\Delta = -q$ are called anti-chiral fields. The same operator product, eq. (38), holds for these with G^- and G^+ interchanged. Thus the singlets in the massless spectrum are in 1–1 correspondence with the chiral primary fields in the theory with charge one. Let $C = \hat{C}\exp(i\phi_i/\sqrt{3})$ be any such chiral field, where $\hat{C}$ is a neutral field. The superpartner of this chiral primary field is then a spinor field of the form,

$$
S\hat{C}e^{-\frac{i\phi_i}{2\sqrt{3}}}. \tag{39}
$$

Similarly, the anti-chiral fields with charge -1 give rise to some singlets, along with anti–spinor fields. It is easy to see that all spinor and anti–spinor fields are given in this form. Thus the spectrum contains 4 dimensional left and right moving fermions along with their scalar superpartners. We conclude that the spectrum of the theory (for a bosonic right moving sector) contains the usual $N = 1$ supergravity multiplet, gauge multiplet and left and right four dimensional chiral fermion multiplets.

.3. Heterotic–like String theory

Let us turn now to the problem of constructing heterotic–like string theories in four dimensions. By heterotic–like we mean a string theory in which the left movers are superstring–like and the right movers are bosonic. Thus also the total trace anomaly of the left movers is $c = 12$ and that of the right movers is $c = 24$. The fact that the theory is inherently left–right asymmetric immediately presents considerable difficulty with modular invariance. The only generic way to satisfy the requirement of modular invariance is to have a left–right symmetric theory. Indeed, in the original construction of the ten–dimensional heterotic string [17] the requirement of modular invariance can be satisfied, rather miraculously, only for the gauge groups $E_8 \times E_8$ and $SO(32)$.

In constructing a heterotic–like string theory in dimension less than 10, using a non–trivial conformal field theory, we encounter even a more serious problem with modular invariance, since general conformal field theories tend to have very restricted possibilities for achieving modular invariance.

The resolution to this problem, as we shall see, is a general map that takes any consistent superstring–like theory to a fully consistent heterotic–like string theory. Since it is easy to construct superstring–like theories, this map provides a universal construction of heterotic string theories.

Consider then a $D = d+2$ dimensional superstring. The left and right movers of the theory correspond to a $c = 12$ conformal field theory which is composed of left and right moving space–bosons, X^μ and $\bar{X}^\mu$, where $\mu = 1, 2, \ldots, d$, left and right moving fermions with space–time degrees of freedom, ψ^μ and $\bar{\psi}^\mu$, and an internal $c = 12 - 3d/2$ conformal field theory with $N = 1$ superconformal invariance. Any such internal conformal field theory would give rise to a fully consistent d dimensional superstring theory.

Consider now the 1–loop partition function of this string theory. When viewed as a two dimensional conformal field theory, the theory is essentially a tensor product of the free fermions and free bosons theories with the internal conformal field theory. The fermions ψ^μ and $\bar{\psi}^\mu$ form a representation of the affine Lie algebras $SO(d)_L \times SO(d)_R$ at level one ($k = 1$). The currents which generate these algebras are as usual $J_{ij} = i{:}\psi^i\psi^j{:}$ and a similar expression for the right movers. This symmetry is, in fact, the remaining part of the Lorentz invariance in the light–cone gauge. Namely, the Lorentz transformations involving only the transverse dimensions. Thus this affine symmetry must be unbroken in the spectrum of the theory or else Lorentz invariance is lost. It follows that the spectrum of the theory must fall into representations of the affine Lie algebra $SO(d)_L \times SO(d)_R$ at level one. For any even d there are four integrable representations of the affine Lie algebra $\hat{SO}(d)$ at level one. These are the singlet, the vector, the spinor and the anti–spinor representations. (More precisely, these are the integrable highest weight representations, but as discussed in ref. [4] these are the only ones that need to concern us.) The 1–loop partition function thus contains the characters of these representations. Define the classical theta functions associated with the affine Lie algebra $\hat{G}$ at level k by,

$$\Theta_\lambda(\tau, z, u) = e^{-2\pi iku} \sum_{\gamma \in M + \lambda/k} e^{\pi ik\tau\gamma^2 - 2\pi ik\gamma z}, \tag{40}$$

where M is the long root lattice of the simple Lie algebra G, and λ is a weight of the algebra which obeys the integrability condition: $k \geq \lambda\theta$, where θ is the highest root normalized to $\theta^2 = 2$. For a simply laced Lie algebra G at level one, the characters are

$$B_\lambda = \frac{\Theta_\lambda(\tau,0,0)}{\eta(\tau)^l}, \tag{41}$$

where the theta functions are taken at level $k=1$ and $\eta(\tau)$ denotes the Dedekind's function and l is the rank of the algebra. Under modular transformations these partition functions form a unitary representation of the modular group. For the Lie algebra $SO(2n)$ this representation is four dimensional. Arrange the four characters of $SO(2n)$ at level one into a vector, $\vec{B} = (B_0, B_v, B_s, B_{\bar{s}})$, where 0 stands for the singlet, v stands for the vector, s and $\bar{s}$ stand for the spinor and anti–spinor representations. Under $S:\tau \to -\frac{1}{\tau}$, we find,

$$\vec{B}(-\frac{1}{\tau}) = S_{2n}\vec{B}(\tau), \qquad \text{where } S_{2n} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix}, \tag{42}$$

and under $T:\tau \to \tau+1$ as

$$\vec{B}(\tau+1) = T_{2n}\vec{B}, \qquad \text{where } T_{2n} = e^{-\pi i n/12}\mathrm{diag}(1,-1,e^{\pi i n/4},e^{\pi i n/4}). \tag{43}$$

Denote by

$$Z_{\lambda,\bar{\lambda}}(\tau) = \mathop{\mathrm{Tr}}_{\mathcal{H}_{\lambda,\bar{\lambda}}} e^{2\pi i\tau(L_0-c/24)}e^{-2\pi i\tau^*(\bar{L}_0-c/24)}, \tag{44}$$

the partition function of the internal $c = 12 - 3d/2$ conformal field theory, where the trace is taken over the representations which are coupled to the λ representation of $\hat{SO}(d)_L$ and $\bar{\lambda}$ representation of $\hat{SO}(d)_R$. The λ and $\bar{\lambda}$ stand for the four integrable representations of $\hat{S}O(d)$. The 1–loop vacuum to vacuum amplitude for the most general superstring can be written as

$$Z(\tau) = \left|(\mathrm{Im}\,\tau)^{-d/2}\eta(\tau)^{-2d}\right| \sum_{\lambda,\bar{\lambda}} B_\lambda(\tau)B_{\bar{\lambda}}(\tau)^* Z_{\lambda,\bar{\lambda}}(\tau), \tag{45}$$

where the first factor is due to the space–time bosons. The consistency of the theory implies, among others, that the total partition function Z is modular invariant.

Let us turn now to the construction of the heterotic–like string theory. For an arbitrary d dimensional string theory, whose partition function is of the form eq. (45), as we shall now show, one obtains a fully consistent heterotic–like string theory in d dimensions.

Consider the representation of the modular group on the $SO(d)$ characters, eqs. (42-43). Now, an important point is that under any modular transformation the $SO(d)$ characters and the $SO(24+d)$ ones transform in the same way. More precisely, the $SO(d)$ and the $SO(24+d)$ characters form isomorphic representations, where the isomorphism exchanges the singlet and vector characters and flips the sign of the spinor and anti–spinor ones. The matrix which implements this change of basis is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{46}$$

The isomorphism is encapsulated in the relations

$$T_d = M T_{24+d} M, \qquad S_d = M S_{24+d} M. \tag{47}$$

Analogous relations hold, with the same change of basis, for the characters of $E_8 \times SO(8+d)$.

From the superstring theory in d dimensions we may form a heterotic string theory by simply replacing the space–time fermions in the right moving sector by internal fermions. To get the correct value for the central charge of the left movers, $c = 24$ we need to replace d fermions with $24+d$ ones. The trace anomaly would then be, $c = d + (12 - 3d/2) + (24+d)/2 = 24$. Of course, if we do this replacement arbitrarily, the resulting heterotic–like string theory would not be consistent. To get a consistent theory we use the isomorphism of the modular transformations of $SO(d)$ and $SO(24+d)$ noted above. If we replace the singlet representation of $SO(d)$ with the vector of $SO(24+d)$ and vice versa, and flip the sign of the spinor representations, then the new partition function, now describing a heterotic–like string theory, would be modular invariant in view of the isomorphism eq. (47). The partition function of the heterotic theory is

$$Z^{\mathrm{het}}(\tau) = \left|(\mathrm{Im}\,\tau)^{-d/2}\eta(\tau)^{-2d}\right| \sum_{\lambda,\bar{\lambda}} B^{\mathrm{SO}}(d)_{\lambda}(\tau)\left(MB^{\mathrm{SO}}(24+d)\right)^{*}_{\bar{\lambda}}(\bar{\tau})Z_{\lambda,\bar{\lambda}}(\tau), \tag{48}$$

The condition of spin–statistics is obeyed in this partition function if it was obeyed in the superstring theory. This condition amounts to the fact that space–time bosons appear with positive multiplicity in the partition function, and space–time fermions appear with negative multiplicity. Spin–statistics is preserved since the signs of the right moving spinor partition functions flip when going from the superstring to the heterotic string, and indeed these states represent space–time bosons in the heterotic string if they were space–time fermions in the superstring, and vice versa.

Consider now the massless content in the right moving sector of the theory. Since the right moving sector describes a bosonic theory, the conformal dimension of all the massless fields in this sector is $\Delta = 1$. As discussed above, the content of this sector is the same as the left moving sector with the exchange of the singlet of $SO(d)$ with the vector of $SO(8+d)\times E_8$ or $SO(24+d)$, and vice versa, and the change of the spinor to the anti–spinor and vice-versa. Let us concentrate here on the physically most interesting case of $E_8 \times SO(8+d)$, especially for $d = 2$ (a four dimensional string theory). We can assign $U(1)$ charges to the representations of the $SO(8+d)$ current algebra since it is a free Majorana fermion system; the $U(1)$ charge is simply the fermion number. The dimensions $\bar{\Delta}_c$ and $U(1)$ charges, $\bar{Q}_c$, for the four representations of $E_8 \times SO(8+d)$ current algebra at level one are listed in table 2.

Table 2.
Dimensions and charges for the right movers.

Representation	$\bar{\Delta}_c$	$\bar{Q}_c$	$\bar{\Delta}_i$	$\bar{Q}_i$
Singlet	0	0	1	$0, \pm 2$
Vector	$\frac{1}{2}$	1	$\frac{1}{2}$	± 1
Spinor	$\frac{5}{8}$	$-\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{2}$
Antispinor	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$-\frac{1}{2}$

Now, according to the rules of replacing the representations of $SO(2)$ with the ones of $E_8 \times SO(10)$ the total $U(1)$ charge of the right movers is an even integer. From this we can compute the dimensions, $\bar{\Delta}_i$, and $U(1)$ charges, $\bar{Q}_i$, of the fields from the internal conformal field theory which couple to each of these representations.

One field which is guaranteed to appear in the internal conformal field theory is the identity field. Let us consider the fields in the right moving sector which couple to the identity field. These are fields of dimension one coming entirely from the transverse bosons or the current algebra $E_8 \times SO(10)$. From the the transverse bosons we have, $\partial_{\bar{z}} X^\mu$. From the gauge group we have the currents, $\bar{J}^a(\bar{z})$ which obey the $E_8 \times SO(10)$ current algebra,

$$\bar{J}^a(\bar{\zeta}) \bar{J}^b(\bar{z}) = \frac{\delta_{ab}}{(\bar{\zeta} - \bar{z})^2} + \frac{f_{abc} \bar{J}^c}{\bar{\zeta} - \bar{z}} + \ldots, \tag{49}$$

where f_{abc} are the structure constants of $E_8 \times SO(10)$. These fields multiply the fields in the left moving sector that couple there to the identity field. As discussed in section (2), these are the vector, the spinor and anti–spinor of $SO(2)$ corresponding to the space–time fermions. The vertex operators for these left movers are ψ^μ, $S^\dagger e^{i\phi}$ and $S e^{-i\phi}$. When multiplying these left movers with the field $\partial_{\bar{z}} X^\nu$ from the right moving sector we get the usual $N = 1$ supergravity multiplet in four dimensions. From the product $\psi^\mu \partial_{\bar{z}} X^\nu$, we get the vertex operators of the graviton (the symmetric part) anti–symmetric tensor field (the anti–symmetric part) and the dilaton (the trace). Multiplying by the $SO(2)$ spinors gives their fermionic superpartners.

Similarly, multiplying these fields with the $E_8 \times SO(10)$ currents, $\bar{J}^a(\bar{z})$, gives the vertex operators of the $E_8 \times SO(10)$ gauge group. The $SO(2)$ spinors give their fermionic superpartners.

The invariance of the right moving sector with respect to the right moving $U(1)$ current algebra, $\partial_{\bar{z}}\bar{\phi}$ and the right moving supersymmetry charges,

$$Q = S_\alpha e^{i\sqrt{3}\bar{\phi}_i/2} \qquad \text{and } Q^\dagger = \bar{S}_\alpha e^{-i\sqrt{3}\bar{\phi}_i/2}, \tag{50}$$

where S_α and $\bar{S}_\alpha$ are the spinor and anti–spinor representations of the $SO(10)$ current algebra, implies that the identity field appears in the same conformal block as these fields. In other words, the left moving $v + s + \bar{s}$ representations of $SO(2)$ multiply the additional right moving fields $\partial_{\bar{z}}\bar{\phi}_i$, Q and $Q^\dagger$. It is easy to see that indeed these fields have the dimension one and charge zero, and thus appear in the massless spectrum. Since these fields depend only on $\bar{z}$, they are anti–holomorphic fields of dimension one. From the associativity of the operator product algebra such fields always correspond to a current algebra for some semi–simple Lie algebra. The Lie algebra generated by the $E_8 \times SO(10)$ currents, $\bar{J}^a$, together with the fields $\partial_{\bar{z}}\bar{\phi}$, Q_α and $Q^\dagger_\beta$ is, in fact, the $E_8 \times E_6$ current algebra at level one. The additional fields complete $SO(10)$ to E_6. The number of fields is 45 (from the adjoint of $SO(10)$) 16 (spinor of $SO(10)$) another 16 (anti–spinor) and 1 singlet, giving a total of 78 currents which is indeed the dimension of E_6. These fields are precisely the ones needed for the vertex operator construction of the $E_8 \times E_6$ current algebra at level one.

To see this consider the root space decomposition of $SO(10)$. The roots of $SO(10)$ are given by $\pm(\epsilon_i \pm \epsilon_j)$ where ϵ_i, $i = 1, 2, \ldots, 5$ are orthogonal unit vectors. To the Cartan subalgebra of $SO(10)$ we add the generator $\partial_{\bar{z}}\bar{\phi}$, the right moving $U(1)$ current, which commutes with the $SO(10)$ currents. Denote the root space vector associated with the $U(1)$ by the unit vector κ. The eigenvalues of this six dimensional Cartan–subalgebra are, $\pm\epsilon_i \pm \epsilon_j$ (from the adjoint of $SO(10)$), and, form Q and $Q^\dagger$,

$$\frac{1}{2}(\delta_1\epsilon_1+\delta_2\epsilon_2+\ldots+\delta_5\epsilon_5)+\frac{\sqrt{3}}{2}\delta_6\kappa \qquad \text{where } \delta_i=\pm 1 \quad \text{and } \delta_1\delta_2\ldots\delta_6=1. \tag{51}$$

These are precisely the roots of E_6. The simple roots are given by $\epsilon_1-\epsilon_2$, $\epsilon_2-\epsilon_3$, $\epsilon_3-\epsilon_4$, $\epsilon_4-\epsilon_5$, $\epsilon_4+\epsilon_5$ and $-\frac{1}{2}(\epsilon_1+\epsilon_2+\ldots+\epsilon_5)+\frac{\sqrt{3}}{2}\kappa$. It is easy to check that all the positive roots are sums of simple roots. Also, the length of each vector is 2 and the scalar products are either 0 or -1. The extra simple root has a scalar product -1 only with the simple root $\epsilon_4+\epsilon_5$. Thus the Dynkin diagram of this algebra is that of $SO(10)$ with an extra node attached at one end. This is precisely the Dynkin diagram of E_6. Thus we proved that the actual gauge symmetry of the theory is $E_8\times E_6$, and that the spectrum contains the gauge multiplet of this gauge group.

The invariance of the entire theory under this gauge group follows from the invariance of the theory under the $U(1)$, as well as the invariance under the supersymmetry charges Q and $Q^\dagger$.

We conclude that from the unit field we get the $N=1$ supergravity multiplet as well as the gauge multiplet of the group $E_8\times E_6$. The same argument applies, mutatis mutandis, in other dimensions. In 6 dimensions the resulting gauge group is $E_8\times E_7$ and in 8 dimensions it is $E_8\times E_8$. We leave the verification of this as an exercise for the reader.

For $N=2$ theories where the algebra is further extended, more fields might appear in the identity conformal block. To appear in the massless sector, such fields must have dimension one and thus would correspond to some additional current algebra symmetry, leading to more gauge bosons. In this case the gauge symmetry would be $E_8\times E_6\times G$ where G is the extra 'enhanced' gauge symmetry. We would see examples of such enhanced gauge symmetries in the sequel. Similarly, more spinor fields might appear in the left moving sector. Such fields would correspond to additional supersymmetries. However, since in four dimensions the existence of more supersymmetries rules out chiral fermions, such theories are not interesting from the phenomenological viewpoint.

Let us consider now the right moving vector representation. From table 2 the internal dimension for the vector fields is $\bar{\Delta}_i = \frac{1}{2}$ and the charge is $\bar{Q}_i = \pm 1$. Again, these fields obey $\bar{\Delta} = |\bar{Q}|/2$, and are thus chiral or anti–chiral fields of the right moving $N = 2$ algebra with charges ± 1. Denote such a field with $\bar{Q}_i = 1$ by $C = \hat{C} \exp(i\bar{\phi}/\sqrt{3})$, where $\hat{C}$ is a neutral field. The vertex operator for the massless fields in the vector representation of $SO(10)$ is $V_\mu \hat{C} \exp(i\bar{\phi}/\sqrt{3})$, where V_μ, $\mu = 1, 2, \ldots 10$ represents the vector of $SO(10)$ at level one. (V_μ can be taken to be 10 free Majorana fermions.) Acting on this field with the right moving supersymmetry generator $Q^\dagger$ we obtain a massless spinor field, $S_\alpha \hat{C} \exp(-i\bar{\phi}/2\sqrt{3})$, where S_α is the spin field of $SO(10)$. Acting once more with $Q^\dagger$ gives the massless singlet field $\hat{C} \exp(-2i\bar{\phi}/\sqrt{3})$. Counting states we find $10 + 16 + 1 = 27$. Indeed these fields together give the 27 representation of E_6. It can be easily checked that the weights of these fields are the correct ones for the 27 of E_6 (as we did for the adjoint), and that this is precisely the vertex operator representation for the 27 of E_6 at level one.

Similarly, the right moving vector fields with $\bar{Q}_i = -1$ give the $\bar{27}$ representation of E_6 when acting with Q twice, $\bar{27} = 10 + \bar{16} + 1$. It can be further seen that these are all the possible fields in the right moving sector of the theory.

How are the right movers in the 27 and $\bar{27}$ representations of E_6 connected together with the right movers? The only possible fields in the right moving sector that these fields can multiply are the spinor and anti–spinor multiplets of $SO(2)$. We thus have four possibilities: space–time left fermions which are 27 ($Q_i = \bar{Q}_i = 1$), right fermions which are 27 ($-Q_i = \bar{Q}_i = 1$), left fermions which are $\bar{27}$ ($Q_i = -\bar{Q}_i = 1$) and right fermions which are $\bar{27}$ ($Q_i = \bar{Q}_i = -1$). The last two are CPT conjugates of the first two ($Q_i \to -Q_i$ along with $\bar{Q}_i \to -\bar{Q}_i$). We conclude that the matter content of the theory consists of a number of left handed fermions in the 27 of E_6 and a number of left–handed fermions in the $\bar{27}$ of E_6. The 27 fields correspond to left and right chiral fields, (c,c), whereas the $\bar{27}$ correspond to the fields which are left chiral and right anti–chiral, (c,a). In general the number of 27 fields, N_{27} would be different from the number of $\bar{27}$, $N_{\bar{27}}$, giving rise to a net number of chiral generations in the theory, $N = N_{27} - N_{\bar{27}}$.

.4. Minimal superconformal field theories

So far we have been discussing the structure of a four dimensional heterotic string theory based on any $N = 2$ superconformal field theory. In order to construct actual examples of such theories we need to find $N = 2$ superconformal field theories. The simplest non–trivial such theories are the so called minimal $N = 2$ theories ref. [8, 7, 9]. These are the only unitary $N = 2$ theories with central charge $c < 3$. The central charge of the k'th minimal model (where k is any positive integer) is

$$c = \frac{3k}{k+2}. \tag{52}$$

There is a simple construction of the $N = 2$ minimal theories [8, 18] by adding one free boson to the Z_k parafermionic field theories [20, 19]. The Z_k parafermionic field theories contain the parafermion ψ_1 and its hermitian conjugate $\psi_1^\dagger$. The fields obey the OPE,

$$\psi_1(z)\psi_1^\dagger(w) = (z-w)^{-2h_1} + \frac{2h_1}{c}(z-w)^{2-2h_1}T_k(w) + \dots \tag{53}$$

where $h_1 = (k-1)/k$ is the dimension of the field ψ_1 and $c = 2(k-1)/(k+2)$ is the central charge of the theory. For every positive integer k there is one such theory. The Z_k parafermionic theories are intimately related with the $SU(2)$ current algebra [20, 19]. The fields of the theory, ϕ^l_m, are labeled by two integers l and m, ϕ^l_m, where l is an isospin index, $0 \le l \le k$ and m relates to the Z_k charge. Similar indices label the right movers, so we will denote the full parafermionic field by $\phi^{l,\bar{l}}_{m,\bar{m}}$. The Z_k charge is given by $(m+\bar{m})/2 \bmod k$. The left dimension of the 'primary' field ϕ^l_m is given by

$$\Delta = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}. \tag{54}$$

when $|m| < l$. The dimensions of the fields $\phi^l_{\pm m}$ and $\phi^{k-l}_{k\pm m}$ are the same. Similar formula holds for the right movers. The field ψ_1 may be identified with $\phi^{0,0}_{2,0}$.

The Z_k charge conservation is consistent with the following operator product,

$$\psi_1(z)\phi^l_m(w) = \sum_{j=-\infty}^{\infty} (z-w)^{-\frac{m}{k}+j-1} A_{(m+1)/k-j}\phi^l_m(w), \tag{55}$$

and

$$\psi_1^\dagger(z)\phi^l_m(w) = \sum_{j=-\infty}^{\infty} (z-w)^{m/k+j-1} A^\dagger_{(1-m)/k-j}\phi^l_m(w), \tag{56}$$

where the A and $A^\dagger$ are operators acting on the field ϕ.

Consider now a conformal field theory composed out of the Z_k parafermionic system and one free boson, $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$. The field $\phi(z,\bar{z})$ obeys

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -2\ln|z-w|.$$

The energy momentum tensor is $T_\phi = -\frac{1}{2}\partial_z\phi\partial_z\phi$. The total energy momentum for such a system is $T = T_\phi + T_k$, and the central charge is $c = 3k/(k+2)$.

In the combined free boson–parafermion system we may construct the fields,

$$\begin{aligned} G^+(z) &= \sqrt{\frac{2k}{k+2}}\psi_1{:}e^{i\phi\sqrt{\frac{k+2}{k}}}{:}, \\ G^-(z) &= \sqrt{\frac{2k}{k+2}}\psi_1^\dagger{:}e^{-i\phi\sqrt{\frac{k+2}{k}}}{:}, \\ J &= i\sqrt{\frac{k}{k+2}}\partial_z\phi. \end{aligned} \tag{57}$$

From eq. (53) it is easy to check the operator product,

$$G^+(z)G^-(w) = \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+J'(w)}{z-w} + \dots \tag{58}$$

which implies the $N=2$ algebra anticommutation relations, eq. (3). Similarly, the other relations in eq. (3) can be verified. It follows that the fields J, $G^\pm$ and T generate an $N=2$ superconformal field theory with the central charge given in eq. (52).

The primary fields of the $N=2$ superconformal algebra are all of the form

$$V = \phi^l_m{:}e^{i\alpha_m\phi(z)}{:}. \tag{59}$$

The fields in the Neveu–Schwarz sector are the ones which are local with respect to $G^\pm$. The primary fields obey

$$G^\pm(z)V(w) = O(\frac{1}{z-w}) + \dots. \tag{60}$$

Similarly, the Ramond sector fields are semi-local with respect to $G^\pm$ (pick up a minus sign when circled around $G^\pm$) and the primary fields are at most singular as $O(\frac{1}{\sqrt{z-w}})$. Using eqs. (55-56) it is easy to see which fields correspond to the primary fields of the Neveu–Schwarz and Ramond sectors. We find,

$$\alpha_m = \frac{m-(\frac{1}{2}\,\mathrm{sign}(0)+a)k}{\sqrt{k(k+2)}}, \qquad m = \dots, l-2, l. \tag{61}$$

$$\alpha_m = \frac{m-(\frac{1}{2}\,\mathrm{sign}(0)+1+a)k}{\sqrt{k(k+2)}}, \qquad m = l, l+2, \dots, \tag{62}$$

where in the Ramond sector $a=0$ and $\mathrm{sign}(0)=\pm 1$ and in the NS sector $a=\frac{1}{2}$ and $\mathrm{sign}(0)=-1$. The conformal dimension of the field V is, $\Delta = \Delta(\phi^l_m)+\alpha_m^2/2$.

The $U(1)$ charge is $\sqrt{\frac{k}{k+2}}\alpha_m$. It follows that the dimensions and $U(1)$ charges in the NS sector are,

$$\Delta = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4(k+2)}, \qquad Q = \frac{m}{k+2}, \tag{63}$$

for $l = 0, 1, \ldots, k$ and $m = -l, -l+2, \ldots, l$. In the Ramond sector the dimensions and charges are

$$\Delta = \frac{l(l+2)}{4(k+2)} - \frac{(m \pm 1)^2}{4(k+2)} + \frac{1}{8}, \qquad Q = \frac{m \pm 1}{k+2} \mp \frac{1}{2}. \tag{64}$$

.5. Partition functions for the minimal theories

Using the connection of the minimal $N = 2$ superconformal field theories with the Z_k parafermions allows us to compute the one loop partition functions of such theories. As in section (2) denote by $\chi^l_m(\tau, z, u)$ the full character of the $N = 2$ algebra over the representation $\mathcal{H}^l_m$ which is defined to be the representation of the $N = 2$ algebra containing the field $\phi^l_m \exp(i\alpha_m \phi)$. For convenience, we shall split each $N = 2$ representation in two, grouping together states related by the action of an even number of $G^\pm$. The field G^+, when it acts on the field $V = \phi^l_m \exp(i\alpha_m \phi)$, produces in the leading term of the operator product the field $\phi^l_{m+2} \exp(i(\alpha_q + \sqrt{(k+2)/k})\phi)$. In addition, we can act arbitrarily with the $U(1)$ current J, implying the $U(1)$ invariance of the spectrum. Also, we can act on the representation with any combination of the parafermions which has a net Z_k charge zero, since all such combinations appear in the operator products of even numbers of $G^\pm$. Thus the representation of the $N = 2$ algebra includes all the fields obtained from V by shifting the Z_k charge and simultaneously shifting the $U(1)$ charge. Applying $2k$ times the field G^+, the parafermionic piece cancels, and we are left with a net shift in the $U(1)$ charge, which shows that the bosonic piece is given by a theta function at level $k(k+2)$. The splitting of the representations into two is most conveniently described by introducing an index s which is defined modulo 4 and is even in the NS sector and odd in the R sector. Denote the partition function of the s sector by $\chi_m^{l(s)}(\tau, z, u)$ defined as in equation (14). Then from the arguments above,

$$\chi_m^{l(s)}(\tau, z, u) = \sum_{j \bmod k} c^l_{m+4j-s}(\tau)\Theta_{2m+(4j-s)(k+2),2k(k+2)}(\tau, 2kz, u), \tag{65}$$

where in the R sector we have replaced $m \to m+s$. c^l_m are the characters of the parafermionic field theory which are given by,

$$c^l_m(\tau) = \eta(\tau)^{-3} \sum_{\substack{-|x|<y\leq|x| \\ (x,y) \text{ or } (\frac{1}{2}-x,\frac{1}{2}+y)\in(\frac{l+1}{2(k+2)},\frac{m}{2k})+Z^2}} \text{sign}(x) e^{2\pi i\tau[(k+2)x^2-ky^2]}, \tag{66}$$

and $\eta(\tau)$ is the Dedekind's function. The classical theta functions of $SU(2)$ at level m are defined by

$$\Theta_{n,m}(\tau, z, u) = e^{-2\pi i u} \sum_{j\in Z+n/2m} e^{2\pi i\tau m j^2 + 2\pi i j z}, \tag{67}$$

where n is defined modulo $2m$. The character $\chi_m^{l(s)}$ is invariant under $m \to m+4$ and $m \to m+2(k+2)$ which shows that indeed s is defined modulo 4 and m is defined modulo $2(k+2)$. Also $\chi_m^{l(s)} = 0$ if $l+m+s \neq 0 \bmod 2$. The character is also invariant under the simultaneous interchange, $l \to k-l$ and $q \to q+k+2$. The action of the modular group on the classical theta functions is given by

$$\Theta_{n,m}(\tau+1, z, u) = e^{\pi i n^2/(2m)}\Theta_{n,m}(\tau, z, u), \tag{68}$$

and

$$\Theta\left(-\frac{1}{\tau}, \frac{z}{\tau}, u+\frac{z^2}{2m\tau}\right) = \frac{1}{\sqrt{2m}}(-i\tau)^{1/2} \sum_{l \bmod 2m} e^{-\pi i l n/m}\Theta_{l,m}(\tau, z, u). \tag{69}$$

From the modular properties of the characters one can prove the identity,

$$\sum_{m \bmod 2(k+2)} \chi_m^{l(s)}(\tau, z, 0)\Theta_{m,k+2}(\tau, -2z, 0) = A^l(\tau, 0, 0)\Theta_{s,2}(\tau, 2z, 0), \tag{70}$$

where A^l are the characters of the $SU(2)$ current algebra at level k,

$$A_l = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}. \tag{71}$$

The identity eq. (71) has a very important consequence. It implies that under modular transformations, the l, m and s indices transform independently, where the l index transforms as a level k $SU(2)$ partitions function, the m index transforms like a level $k+2$ theta function and the s index transforms as a level 2 theta function. Explicitly,

$$\chi_m^{l(s)}(-\frac{1}{\tau}, 0, 0) = C \sum_{\substack{l', m', s' \\ l'+m'+s'=0 \bmod 2}} \sin\left(\pi \frac{(l+1)(l'+1)}{k+2}\right) e^{\frac{\pi i m m'}{k+2}} e^{-\frac{\pi i s s'}{2}} \chi_{m'}^{l'(s')}(\tau, 0, 0). \tag{72}$$

and

$$\chi_m^{l(s)}(\tau + 1, 0, 0) = e^{2\pi i \gamma_m^{l(s)}} \chi_m^{l(s)}(\tau, 0, 0), \tag{73}$$

where

$$\gamma_m^{l(s)} = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4(k+2)} + \frac{s^2}{8}. \tag{74}$$

which follows from the modular transformations of the theta functions, eqs. (68-69). C is some constant determined by unitarity. From the modular properties of the theta function it can be seen that, indeed, the full character transforms according to eq. (19). We leave this as an exercise for the reader.

The fact that the modular transformations of the $N = 2$ characters factorize in the way described above implies that we can write a modular invariant partition function for the minimal theories starting from any modular invariant of the $SU(2)$ current algebra, and the two theta function systems. Take $Z_A = \sum_{l,\bar{l}} N_{l,\bar{l}} A^l A^{\bar{l}*}$, $Z_{k+2} = |\eta|^{-2} \sum_{m,\bar{m}} L_{m,\bar{m}} \Theta_{m,k+2} \Theta^*_{m,k+2}$ and

Table 3.

The $SU(2)$ current algebra partition functions[21].

$SU(2)$	$\sum_{l=1}^{k+1} A_l A_l^*$	$k \geq 1$
$SO(3)$	$\sum_{l \text{ odd}=1}^{2j-1} \|A_l + A_{4j+2-l}\|^2 + 2\|A_{2j+1}\|^2$	$k = 4j$
$SO(3)$	$\sum_{l \text{ odd}=1}^{4j-1} \|A_l\|^2 + \sum_{l \text{ even}=2}^{4j-2} A_l A_{4j-l}^*$	$k = 4j - 2,\ j \geq 2$
S_{10}	$\|A_1 + A_7\|^2 + \|A_4 + A_8\|^2 + \|A_5 + A_{11}\|^2$	$k = 10$
S_{16}	$\|A_1 + A_{17}\|^2 + \|A_5 + A_{13}\|^2 + \|A_7 + A_{11}\|^2$ $+\|A_9\|^2 + (A_3 + A_{15})A_9^* + \text{c.c}$	$k = 16$
S_{28}	$\|A_1 + A_{11}\|^2 + \|A_{19} + A_{29}\|^2$ $+\|A_7 + A_{13} + A_{17} + A_{23}\|^2$	$k = 28$

$Z_2 = \sum_{s,\bar{s}} S_{s,\bar{s}} \Theta_{s,2} \Theta^*_{\bar{s},2}$ to be any modular invariant partition functions for the affine $SU(2)$ and theta function systems. Then, a modular invariant partition function for the $N = 2$ minimal theories is given by,

$$W = \frac{1}{2} \sum_{\substack{l,m,s \\ \bar{l},\bar{m},\bar{s}}} N_{l,\bar{l}} L_{m,\bar{m}} S_{s,\bar{s}} \chi_m^{l(s)} \chi_{\bar{m}}^{\bar{l}(\bar{s})^*} . \tag{75}$$

The problem of modular invariance for the $SU(2)$ current algebra was introduced in ref. [4]. A level by level classification of the partition functions, via a direct decomposition of the relevant representations of the modular group, was described in [21]. These partition functions are listed in table 3. They include two infinite sequences corresponding to the $SO(3)$ and $SU(2)$ WZW field theories, along with three sporadic solutions at levels $k = 10, 16, 28$ ‡.

‡ The $k = 28$ solution was later described in ref. [22]. A theorem giving the complete affine $SU(2)$ modular invariants was proved in [19].

The complete list of modular invariant partition functions for the theta function system at level m is given by a theorem proved in ref. [19]. These include the left–right symmetric partition function $Z = \sum_{n \bmod 2m} \Theta_{n,m}\Theta^*_{n,m}$ and its projections by an arbitrary subgroup of the Z_m symmetry group. Thus the complete list of partition functions for the $N=2$ minimal theories is obtained by choosing arbitrary modular invariants for the affine and theta systems. In particular, this analysis shows that the k'th minimal theory has a Z_{k+2} discrete symmetry.

Our next question is to identify the chiral fields for a given $N=2$ minimal theory. These are the fields in the NS sector for which the dimension Δ and the $U(1)$ charge Q obey, $\Delta = Q/2$. From eq. (64) it is easy to check that the only such possible fields are $\phi^{l,\bar{l}}_{l,\bar{l}} e^{i(l\phi+\bar{l}\bar{\phi})/\sqrt{k(k+2)}}$ for some l and $\bar{l}$. Similarly, the anti–chiral fields, which obey $\Delta = -Q/2$, are given by $\Phi^{l,\bar{l}}_{-l,-\bar{l}} e^{-i(l\phi+\bar{l}\bar{\phi})/\sqrt{k(k+2)}}$. Whether any of these fields appears in the spectrum depends on the particular modular invariant chosen. Choosing the theta function invariants to be the diagonal ones, and the $SU(2)$ affine invariant, implies that there is precisely one such field for each value of $l = \bar{l}$, $0 \le l \le k$, a total of $k+1$ fields. Similarly, chosing any of the other invariant gives a chiral field for each left–right symmetric term in the affine partition function. Thus for example, for the S_{10} solution the allowed values of $l = \bar{l}$ are $1, 4, 5, 7, 8, 11$.

.6. Minimal string theories

We can get a $D = d+2$ ($D =$ even) dimensional superstring–like string theory using the minimal models as building blocks. We take a collection of minimal models, $k_1, k_2, \ldots, k_r$ with the total central charge,

$$\sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 12 - \frac{3d}{2}, \tag{76}$$

and joining to it d free bosons, X^μ, and d free fermions, ψ^μ, which represent the Minkowski space degrees of freedom (including both the left and right movers).

The partition function for a given minimal model is given by

$$Z_i = \frac{1}{2} \sum_{l,\bar{l},q,s} N_{l,\bar{l}} \chi_q^{l(s)} \chi_q^{\bar{l}(s)^*}, \tag{77}$$

where $N_{l,\bar{l}}$ is one of the $SU(2)$ affine modular invariants (table 3). We assume here the left–right symmetric modular invariants for the q and s indices.

The fermions with the space–time indices form a $SO(d)$ current algebra. Denoting by Θ_λ the level one theta functions of the algebra (λ ranges over the singlet, the vector or the two spinor representations) the partition function of the current algebra part is,

$$Z_c = \sum_\lambda B_\lambda B_\lambda^*. \tag{78}$$

To form a consistent string theory in this way we cannot simply multiply the partition functions of each sub–theory. The reason is that we must preserve the $N = 1$ supersymmetry on the world sheet, since it is local in the string. To do so we must make sure that the Neveu–Schwarz states in each of the sub–theories will be coupled only to one another and would not mix with Ramond states. Similarly, the Ramond states should couple only to each other.

We may rewrite the partition function Z_i, eq. (77), as follows. Define

$$NS^{\pm} = \frac{1}{2} \sum_{l,\bar{l},q} N_{l,\bar{l}} (\chi_q^{l(0)} \pm \chi_q^{l(2)})(\chi_q^{\bar{l}(0)} \pm \chi_q^{\bar{l}(2)})^*, \tag{79}$$

$$R^{\pm} = \frac{1}{2} \sum_{l,\bar{l},q} (\chi_q^{l(1)} \pm \chi_q^{l(3)})(\chi_q^{\bar{l}(1)} \pm \chi_q^{\bar{l}(3)})^*. \tag{80}$$

The partition functions $NS^{\pm}$ and $R^{\pm}$ may be interpreted as the partition functions on the torus with boundary conditions in the time direction which are ± 1. Under the modular transformation $\tau \to \tau + 1$ we have, $NS^+ \to NS^- \to NS^+$,

$R^+ \to R^- \to R^+$. Under $S : \tau \to -\frac{1}{\tau}$ we have, $NS^+ \to NS^+$, $NS^- \to R^+$, $R^+ \to NS^-$ and $R^- \to R^-$. This implies that the partition function,

$$Z_i = \frac{1}{2}(NS^+ + NS^- + R^+ + R^-) \tag{81}$$

is modular invariant. Indeed this partition is identical to the one given in eq. (77).

We can form similar combinations for any $N = 2$ theory (actually, $N = 1$ supersymmetry is sufficient), which would then transform in the same way. Note that the second terms in eqs. (79-80) correspond to the action of G on the first terms. Thus the generalization of these equations is,

$$NS^{\pm} = \sum_{pq} (\chi_p \pm G(\chi_p))(\chi_q \pm G(\chi_q))^*, \tag{82}$$

where p and q range over the representations in the Neveu–Schwarz sector and each representation is obtained by the action of an even number of $G^{\pm}$. Similarly we can define $R^{\pm}$ as the same sum over the R sector representations. In particular, for the $SO(d)$ current algebra, we can form $NS^{\pm} = |B_0 \pm B_v|^2$ and $R^{\pm} = |B_s \pm B_{\bar{s}}|^2$.

It is now evident how to form a partition function for a product of n theories in which the NS states are coupled only to each other and similarly the R states. Define

$$NS^{\pm} = \prod_i NS_i^{\pm}, \qquad R^{\pm} = \prod_i R_i^{\pm}, \tag{83}$$

where $NS_i^{\pm}$ and $R_i^{\pm}$ are the partition functions for the i'th sub–theory. Then the full partition function,

$$Z = \frac{1}{2}(NS^+ + NS^- + R^+ + R^-) \tag{84}$$

is clearly modular invariant. In addition, the condition of coupling NS only to NS and R only to R is obeyed.

It would be useful to introduce some notation for products of characters of the minimal models. Let us group the l, q and s indices into vectors $\vec{l} = (l_1, l_2, \ldots, l_r)$ and $\vec{v} = (s_0, s_1, s_2, \ldots, s_r, q_1, q_2, \ldots, q_r)$, where $s_0 = v_0$ is a weight of $SO(d)$ at level one and q_i and $s_i,\ i \geq 1$ are q the s indices of the various minimal models. Define the character,

$$Z_{\vec{v}}^{\vec{l}} = B_{v_0} \prod_{i=1}^{r} \chi_{v_{i+r}}^{l_i(v_i)}, \tag{85}$$

which represents a product character for the various theories. Define also

$$N_{\vec{l},\vec{\bar{l}}} = \prod_{i=1}^{r} N_{l_i,\bar{l}_i}^{(k_i)}, \tag{86}$$

the product of multiplicities for each of the $SU(2)$ invariants. We can rewrite the partition function Z, eq. (84), as

$$Z = \frac{1}{2^r} \sum_{l,\vec{l},\vec{v},\vec{\mu}} N_{\vec{l},\vec{\bar{l}}} Z_{\vec{v}}^{\vec{l}} Z_{\vec{v}+\vec{\mu}}^{\vec{\bar{l}}*}, \tag{87}$$

where the sum over the vector μ ranges over all the elements in the lattice spanned by the vectors

$$\vec{\beta}_i = (v \text{ on } 0,\ 2 \text{ on } i,\ 0 \text{ elsewhere}), \qquad \text{for } i = 1, 2, \ldots, r, \tag{88}$$

and the vector $\vec{v}$ obeys $\vec{v}_0 = \vec{v}_i \bmod 2$ for $i = 1, 2, \ldots, r$.

This partition function corresponds to a consistent four dimensional string theory. However it is not supersymmetric. To get supersymmetry we need to use the general supersymmetry projection described in section (2). Define the vector

$$\vec{\beta}_0 = (s, 1, 1, 1, \ldots, 1). \tag{89}$$

It can be seen that the operation of acting with $Q = \exp(i\phi)$ on the minimal characters is equivalent to shifting the vector $\vec{v}$ by $\vec{\beta}_0$. Thus, as described in section (2), we get a supersymmetric partition function by summing over states related by the action of $\vec{\beta}_0$ and in addition eliminating any states in the spectrum for which the left and right total $U(1)$ charges are not odd integers.

To get a heterotic–like string theory we use the map from superstring–like theories into heterotic–like strings described in section (3). This is implemented by replacing the characters of $SO(d)$ in the right sector by the characters of $E_8 \times SO(8+d)$ or $SO(24+d)$. The full partition function for the supersymmetric heterotic–like string theory (without the contribution of the transverse bosons) is then,

$$W = \frac{1}{2^r} \sum_{\substack{\vec{v},\vec{\bar{v}},\vec{l},\vec{\bar{l}} \\ \vec{v}-\vec{\bar{v}} \in Q}} \pm N_{\vec{l},\vec{\bar{l}}} Z^{\vec{l}}_{\vec{v}} Z^{\vec{\bar{l}}*}_{\mu+\vec{\bar{v}}}, \tag{90}$$

where μ is the vector, $\mu = (v \text{ on } 0,\ 0 \text{ elsewhere})$ (implementing the change of $SO(d)$ representations to $SO(8+d) \times E_8$ ones; for the right movers, B stands for the characters of this group), the sign is determined by spin–statistics, Q is the lattice spanned by the vectors $\vec{\beta}_i$ and $\vec{\beta}_0$ and the sum is limited to representations for which the left and right $U(1)$ charges are odd integers, and which are all either in the R sector or all in the NS sector. This partition function represents a fully consistent, modular invariant space–time supersymmetric heterotic–like string theory.

.7. Massless fields in the minimal string theories and manifolds

Let us turn now to the discussion of the explicit spectrum of the minimal string theories described in the previous section. As discussed in sections (2–3) the massless spectrum of any $N=2$ string theory (without enhanced supersymmetry) contains the usual $N=1$ supergravity multiplet. In four dimensions, in addition, there are the gauge bosons for the gauge group $E_8 \times E_6 \times G$ where G is a possible enhanced symmetry group. The rest of the spectrum consists of a number of chiral fermions in the 27 representation of E_6 ('generations'), a number of chiral fermions in the $\bar{27}$ representation of E_6 ('anti–generations') and a number of E_6 singlets, along with their superpartners. The matter fields are all singlets of E_8 but can transform non–trivially under the enhanced gauge group.

Particular string theories might also have discrete symmetries. For example, the k'th minimal model has a discrete symmetry group which is Z_{k+2}. The charge of the field $\phi^{l,\bar{l}}_{q,\bar{q}}$ (in the NS sector) is given by $(q+\bar{q})/2 \bmod k+2$. Denote by $k_1 k_2 \ldots k_r$ the string theory made from the k_1, k_2, to the k_r minimal theories. The discrete symmetry group of such a product is $Z_{k_1+2} \times Z_{k_2+2} \times \ldots Z_{k_r+2}$. Actually, since the discrete symmetry is embedded in the $U(1)$ charge, and the total $U(1)$ charge is an odd integer, the element $g = \{1, 1, \ldots, 1\} \in G$ acts trivially on all the fields in the theory. Thus the actual symmetry group of the product of minimal theories is $G/(g)$, the quotient group of G by the subgroup generated by g. Denote by $(p_1, p_2, \ldots, p_r)$ the charge of a field transforming as $V \to \exp(2\pi i p_n s_n/(k_n+2))V$ under the element $\{s_n\} \in G$. In addition, if a given minimal model appears more than once in the product, we have the freedom to permute the various copies, giving rise to permutation symmetries in the spectrum.

Not all the discrete symmetries commute with the supersymmetry generator. The condition for the element $\{s_1, s_2, \ldots, s_r\}$ to commute with supersymmetry is

$$\sum_{i=1}^{r} \frac{s_i}{k_i+2} = \text{integer}. \tag{91}$$

Other generators of G are R symmetries (i.e., different fields in a supersymmetry multiplet transform differently under these). The odd permutations are also R symmetries.

The generations in a given $N=2$ string theory correspond to the chiral fields of the internal conformal field theory (section 3). Similarly, the anti–generations come from the fields of the type (c, a) (left chiral and right anti–chiral). The chiral fields in the supersymmetric string theory are of two types: 1) Fields which are chiral fields in the original $N=2$ theory before the projection . Any chiral field in the original theory with left and right charges equal to one gives rise to one such field in the string theory. 2) Fields which arise from the supersymmetry projection. The chiral fields in the minimal model are given by fields of the type $x_l = \phi_{l,l}^{l,l} e^{il(\phi+\bar{\phi})/\sqrt{k(k+2)}}$ where $0 \leq l \leq k$ and l appears in the $SU(2)$ modular invariant chosen. There are no fields of the type (c,a) or (a,c) in the minimal models. The chiral fields of charge one in the product of the $k_1, k_2, \ldots, k_r$ theories are thus given by $x_{l_1}^{(1)} x_{l_2}^{(2)} \ldots x_{l_r}^{(r)}$ where the l_i obey $0 \leq l_i \leq k_i$ and

$$\sum_{i=1}^{r} \frac{l_i}{k_i+2} = 1. \tag{92}$$

The multiplicity of this field is $N_{\vec{l},\vec{l}}$ where N is the affine modular invariant chosen (in the notation of section (6)). The discrete symmetry charge of these generations is $(l_1, l_2, \ldots, l_r)$.

The full massless spectrum of a given minimal conformal field theory may be computed by expanding the partition function eq. (90). Such an enumeration of states can be quite tedious to perform by hand, but can easily be computerized. Let us first consider in detail one example. Take the theory obtained from $k+2$ copies of the k'th minimal theory with the $SU(2)$ modular invariant. The central charge of this theory is $(k+2)\frac{3k}{k+2} = 3k$. Thus this theory can be used to compactify the string to $10-2k$ dimensions. The case of $k=3$ is particularly interesting since it gives a compactification to 4 dimensions.

Enumerating states for the theory 3^5 we find that the generations are all given by fields of the type $x_{l_1} x_{l_2} \ldots x_{l_r}$ which obey eq. (92). (This is actually true for any k). There is one anti–generation and 330 E_6 singlets. The gauge group of this theory is $E_8 \times E_6 \times U(1)^4$.

Consider the manifold M_k (Fermat surface),

$$V(Z_i) = Z_1^{k+2} + Z_2^{k+2} + \ldots Z_{k+2}^{k+2} = 0, \tag{93}$$

where the Z_i are complex variables, modulo the identification of fields $\{Z_i\} \equiv \{wZ_i\}$ where w is any complex number. This is a hyper-surface in CP^{k+1}. The complex dimension of this manifold is k. The first Chern class of this manifold vanishes as can be seen by writing the holomorphic $(k,0)$ form on this surface

$$\Omega = \oint \epsilon_{i_1,i_2,\ldots,i_{k+2}} \frac{Z_{i_1} \mathrm{d}Z_{i_2} \mathrm{d}Z_{i_3} \ldots \mathrm{d}Z_{i_{k+2}}}{V(Z_i)}. \tag{94}$$

Note that this form is well defined in CP^k. The case of $k = 1$ corresponds to a torus (the only complex curve with vanishing first Chern class) in the shape of the $SU(3)$ maximal torus. For $k = 2$ this is the maximally symmetric shape of the K_3 manifold. M_3 is the quintic hypersurface in CP^4.

The discrete symmetry group of the manifold M_k is as follows. We may first multiply any of the Z_i by a complex phase which is a $k+2$ root of unity, $Z_i \to w_i Z_i$ where $w_i^{k+2} = 1$. Thus there is a Z_{k+2}^{k+2} symmetry on the surface. Due to the CP^k identification, however, the overall phase is irrelevant. Second, we may permute any of the variables Z_i into one another, $Z_i \to Z_{p(i)}$ where $p \in S_{k+2}$ is any permutation. We conclude that the symmetry group of the manifold M_k is

$$G_k = \frac{S_{k+2} \ltimes Z_{k+2}^{k+2}}{Z_{k+2}}. \tag{95}$$

The Euler number, χ_k, of the manifold M_k can be easily computed by the method (for example) of multiple covers of CP^n. We find $\chi_1 = 0$, $\chi_2 = 24$, $\chi_3 = -200$, etc.

Note that the discrete symmetry group of the string theory k^{k+2} is precisely the same as that of the manifold M_k. Thus, it is not unlikely that the two are related.

The compactification of the field theory limit of the heterotic string on manifolds of the type $M \times K$, where M is a four dimensional manifold and K is a six dimensional manifold, was considered by Candelas et al. [10]. By requiring unbroken $N = 1$ supersymmetry in four dimension, these authors showed that the field equations lead to a three dimensional complex manifold with a Ricci flat metric. Such manifolds are called Calabi–Yau manifolds. The only topological obstruction to finding such a metric is the vanishing of the first Chern class [23]. Thus, for example, the manifold M_3 admits such a metric.

As discussed by Candelas et al. the gauge group in the field theory is $E_8 \times E_6$ with a number of generations and anti–generations in the 27 and $\bar{27}$ of E_6. The number of generations is equal to the number of harmonic $(2,1)$ forms, $h^{2,1}$, and the number of anti–generations is equal to the number of harmonic $(1,1)$ forms, $h^{1,1}$. For the manifold M_3 these numbers are $h^{2,1} = 101$ and $h^{1,1} = 1$. These are precisely the number of generations and anti–generations in the string theory 3^5! There are 102 E_6 singlets in the field theory limit which implement the change of radii and complex structures. In addition, in the field theory there are 224 singlets associated with $H^1(\mathrm{End}\, T)$, which come from an index theorem for the octet of the $SU(3)$ holonomy [24]. Thus, for a field theory compactification on the manifold M_3 we would expect a total of 326 singlets, which is 4 less of what we find for the string theory 3^5. Recall, however, that the theory 3^5 has the enhanced gauge symmetry $U(1)^4$. Thus there must be additional 4 'Higgs bosons' E_6 singlets in the spectrum needed to give the $U(1)$ gauge fields a mass when deforming this theory. Thus the total number of singlets expected from field theory considerations is indeed 330, precisely the number found in the theory 3^5.

The 101 harmonic $(2,1)$ forms (which give the generations in the field theory) correspond to polynomials which can be added to the defining equation (93). These are polynomials of the type $Z_1^{r_1} Z_2^{r_2} \ldots Z_5^{r_5}$ where $0 \le r_i \le 3$ and $\sum_i r_i = 5$. The discrete symmetry charge of this polynomial is $(r_1, r_2, \ldots, r_5)$. These polynomials match precisely the generations in the string theory 3^5. From equation (92) these are given by the fields $x_{r_1} x_{r_2} \ldots x_{r_5}$ where $0 \le r_i \le 3$ and $\sum r_i = 5$. The discrete symmetry charge of this field is $(r_1, r_2, \ldots, r_5)$. Thus, not only the number of generations match precisely, but their charges under all the discrete symmetries are the same in the string theory and on the manifold.

Similarly, it can be seen that the charges for the anti–generations and the

E_6 singlets are precisely the correct ones. There is an isomorphism of the entire massless spectrum in the theory 3^5 into their corresponding polynomials and elements of $H^1(\mathrm{End}\, T)$ (which can be represented by some tensors) which preserves the 75,000 element discrete symmetry group. The singlets along with their isomorphic polynomials and tensors are shown below

(20)	$(0,8,0)\ (0,0,0)^3\ (2,2,0)$	$Z_1^3 Z_5^2$
(30)	$(0,8,0)(0,0,0)^2\ (1,1,0)^2$	$Z_1^3 Z_4 Z_5$
(20)	$(0,0,0)^3\ (1,7,0)\ (3,3,0)$	P_{44555}
(30×2)	$(0,0,0)^2\ (1,1,0)\ (1,7,0)\ (2,2,0)$	$P_{43455}\ P_{53445}$
(30×2)	$(0,0,0)^2\ (1,1,0)\ (2,6,0)\ (3,3,0)$	$P_{34555}\ P_{43555}$
(30)	$(0,0,0)^2\ (2,6,0)\ (2,2,0)^2$	$Z_3 Z_4^2 Z_5^2$
(20)	$(0,0,0)\ (1,1,0)^3\ (1,7,0)$	$Z_2 Z_3 Z_4 Z_5^2$
(20×3)	$(0,0,0)\ (1,1,0)^2\ (2,6,0)\ (2,2,0)$	$P_{23455}\ P_{32455}\ P_{42355}$
(20)	$(1,9,0)^3\ (2,0,0)\ (3,3,0)$	P_{45555}
(5)	$(1,1,0)^4(2,6,0)$	$Z_1 Z_2 Z_3 Z_4 Z_5\ P_{12345}\ P_{21345}\ P_{31245}\ P_{41235}$
(5)	$(1,9,0)^4\ (2,4,0)$	Kahler + 4 extra

(These are the (l,q,s) numbers for the right movers in each of the sub–theories. The numbers above correspond to spinors which are singlets of $SO(10)$.)

We are thus bound to conclude that the string theory 3^5 describes string propagation on the manifold M_3. Similarly, it can be seen that the discrete symmetries and spectra of the k^{k+2} theory are the same as the ones expected for the manifold M^k.

Many other examples of such identifications can be made. One example of physical interest is the string theory $1^1 16^3$ with the S_{16} modular invariant used for all the $k = 16$ theories. This string theory has 35 generations and 9 anti–generations. The manifold which fits the massless spectra of this theory is the hypersurface

$$\begin{aligned} Z_0^3 + Z_1^3 + Z_2^3 + Z_4^3 = 0, \\ Z_1 X_1^3 + Z_2 X_2^3 + Z_3 X_3^3 = 0, \end{aligned} \tag{96}$$

in $CP^4 \times CP^3$, as can be seen by an analysis analogous to the one described above [25]. This theory leads to a three generations model via a quotient with a $Z_3 \times Z_3$ symmetry. Recent study [26] shows that this theory has excellent phenomenological prospects.

The reader might wonder, are all the $N = 2$ string theories in correspondence with manifolds of vanishing first Chern class? We shall argue that the answer to this question is yes [12].

To see this it is useful to first consider compactifications to 8 and 6 dimensions.

To compactify down to 8 dimensions we use an $N = 2$ superconformal field theory with the central charge $c = 3$. We will prove that all $N = 2$ superconformal field theories with central charge $c = 3$ and integral $U(1)$ charges are superconformal field theories on a torus. Namely, they can be realized as a theory of one complex boson and one complex fermion on some even self–dual lattice. Indeed the torus is the only manifold of vanishing first Chern class in one complex dimension.

Denote by ϕ the $U(1)$ free boson, $J = i\partial_z \phi(z)$. Similarly the right current is $\bar{J} = i\partial_{\bar{z}} \phi$. Now, since $G_\pm$ has charges plus and minus one they can be written as

$$G_\pm = \hat{G}_\pm{:}\exp(i\phi){:}, \tag{97}$$

where $\hat{G_\pm}$ are some fields which commute with H. In the case of $c = 3$ the fields $\hat{G}_\pm$ can be represented as one complex boson and its complex conjugate. This follows from the OPE of G_+ with G_-, eq. (58). For $\hat{G_\pm}$ this OPE translates into,

$$\hat{G}_+(z)\hat{G}_-(w) = \frac{2}{(z-w)} + \dots, \tag{98}$$

and similarly $\hat{G}_\pm(z)\hat{G}_\pm(w) =$ regular. These are precisely the OPE relations of a complex free boson H. Using also the holomorphicity of these fields, we can identify

$$\hat{G}_+ = \sqrt{2}\partial_z H, \qquad \text{and} \;\; G_- = \sqrt{2}\partial_z H^\dagger. \tag{99}$$

Now, the free boson ϕ can be fermionized into one complex fermion, $\psi = \,:\!\exp i\phi\!:$ and its complex conjugate (neglecting cocycle factors). Thus, $G^+ = \sqrt{2}\psi\partial_z H$ and $G^- = \sqrt{2}\psi^*\partial_z H^*$. The $U(1)$ current is then $J = i\partial_z\phi = \psi^\dagger\psi$. Finally, the stress energy tensor can be found from the OPE of G^+ and G^-, eq. (58),

$$T = \frac{1}{2}\psi^\dagger\partial_z\psi - \frac{1}{2}\partial_z H^\dagger\partial_z H. \tag{100}$$

This is the canonical form of the $N = 2$ algebra of a theory of one complex fermion and one complex boson. In other words, the $N = 2$ superconformal algebra at $c = 3$ is identical to the algebra of one complex boson and one complex fermion after a simple change of variables.

Thus any such conformal field theory must contain these fields. What about the rest of the fields in the theory? Assuming that the $U(1)$ charges are all integral means that each field in the theory has an integral fermion number and thus can be expressed in terms of these free fermions and free bosons. (In other words, the free boson ϕ lives precisely at the radius one which can be fermionized.) All that remains is to specify the values of the zero modes (momenta) for the complex free boson. Due to the closure of the operator product algebra these momenta form an additive group – a lattice. Due to modular invariance this lattice must be an even self dual Lorenzian lattice with signature (2,2) [1]. To summarize, we proved that any $c = 3$ $N = 2$ superconformal field theory with integral $U(1)$ charges is equivalent to some Narain type toroidal theory on a $(2,2)$ lattice.

Table 4.

Theory	s in 56	$\bar{s}$ in 56	E_7 singlets	Gauge
1^6	20	0	140	$\mathrm{U}(1)^5$
$1^4 4^1$	20	0	140	$\mathrm{U}(1)^5$
$1^2 2^1 10^1$	20	0	136	$\mathrm{U}(1)^3$
$1^2 4^2$	20	0	136	$\mathrm{U}(1)^3$
$1^1 2^2 4^1$	20	0	146	$\mathrm{SU}(2)^2 \times \mathrm{U}(1)^2$
$1^1 5^1 40^1$	20	0	134	$\mathrm{U}(1)^2$
$1^1 6^1 22^1$	20	0	134	$\mathrm{U}(1)^2$
$1^1 7^1 16^1$	20	0	134	$\mathrm{U}(1)^2$
$1^1 8^1 13^1$	20	0	134	$\mathrm{U}(1)^2$
$1^1 10^2$	20	0	134	$\mathrm{U}(1)^2$
2^4	20	0	136	$\mathrm{U}(1)^3$
$2^1 3^1 18^1$	20	0	134	$\mathrm{U}(1)^2$
$2^1 4^1 10^1$	20	0	134	$\mathrm{U}(1)^2$
$2^1 6^2$	20	0	134	$\mathrm{U}(1)^2$
$3^2 8^1$	20	0	134	$\mathrm{U}(1)^2$
4^3	20	0	134	$\mathrm{U}(1)^2$

From the general theorem described above, it follows that, in particular, when we tensor $N = 2$ minimal models to get a $c = 3$ theory and impose the general supersymmetry projection, the resulting theory describes string propagation on a torus. There are three such possibilities, 1^3, 2^2 and $1^1 4^1$. The first and the last correspond to the $SU(3)$ torus and the 2^2 theory corresponds to the $SU(2)^2$ torus.

Consider now the case of compactification down to 6 dimensions. In this case we need a conformal field theory with $c = 6$. There are 17 combinations of central charges from the discrete series that give this value (using only the left–right symmetric $SU(2)$ and affine modular invariants). One theory, $1^3 2^2$ can be seen to correspond to string propagation on the $SU(3) \times SU(2)^2$ torus. The spectra of the other 16 theories are listed in table 4.

From table (4) we see that all these theories have 20 spinors in the 56 of E_7, no anti–spinors and a number of E_7 singlets equal to 130 plus twice the dimension of the enhanced gauge symmetry group. What is the explanation for this? The only two dimensional complex manifolds of vanishing first Chern class are 4-tori, T^4, and the K_3 manifold. The number of generation expected for a field theory compactification on K_3 is equal to the number of harmonic $(1,1)$ forms, $h^{1,1} = 20$, giving indeed 20 generations. The number of singlets expected is 130 where 40 come from deformations of the complex structure and radii and 90 come from $H^1(\mathrm{End}\, T)$. The extra gauge bosons are Higgs singlets. Thus the numbers in this table match precisely the spectrum expected for the K_3 manifold. We conclude that all these theories correspond to string propagation on the K_3 surface. Since, the theories made from various minimal models along with their very large number of possible projections are not in the least special in the space of $N = 2$ conformal field theories, we must conclude that any $N = 2$ string theory in 6 dimensions describes string propagation on a manifold of vanishing first Chern class.

Finally, it can be seen by various indirect methods that the four dimensional $N = 2$ string theories also correspond to string propagation on manifolds of vanishing first Chern class. For example, there is always a 1-1 map from 27's and $\bar{27}$ in the spectrum to the singlets. This map preserves all the discrete symmetries [27] for any $N = 2$ string theory (not necessarily a minimal one). The map gives the singlets associated with the change of radii and complex structures. Since the harmonic $(2,1)$ and $(1,1)$ forms give both the generations and anti–generations and the singlets that deform the complex structure and radii, such a map is indeed expected for a string theory on a Calabi–Yau manifold. It can be seen that the potential for such singlets is perturbatively flat, as it should be [28].

Another piece of evidence comes from the study of projections of the minimal string theories. If the theory corresponds to string propagation on the manifold M, the theory projected by the discrete symmetry group Z would describe string propagation on the quotient manifold M/Z, where we identify points related by the action of Z on the manifold. Let us digress on this.

In the first step we would like to suggest a formula for the Euler number of a

Calabi–Yau manifold, H, obtained through a quotient of some other manifold, G, by some discrete automorphism group, Z, acting on G. In other words, $H = G/Z$. The interesting case is when the group Z does not act freely on the manifold. In this case H is not a smooth manifold, but rather has some singularities. In many cases of interest, these singularities can be resolved by cutting some small disks around them, and then gluing some smooth non–compact manifolds.

First, we suggest a criteria when this can be done. The singularities can be resolved, and the resulting manifold, $\tilde{H}$ is a smooth manifold of $SU(n)$ holonomy, if and only if the group Z acts trivially on the holomorphic $(n, 0)$ form.

Next we give a formula for the Euler number of $\tilde{Z}$,

$$\chi(\tilde{H}) = \frac{1}{|Z|} \sum_{g,h \in Z} \chi(g, h), \tag{101}$$

where $|Z|$ is the order of Z, and $\chi(g, h)$ is the Euler number of the points of M left fixed by both elements g and h. Eq. (101) was described in ref. [2], for Calabi–Yau manifolds obtained by blowing up orbifolds. We suggest, that it is valid for all manifolds of vanishing first Chern class and in particular all Calabi–Yau manifolds. A variant of eq. (101) gives the Euler number of the manifold H before the resolution of singularities. In the case of Calabi–Yau manifolds, this number is defined to be, $\chi(H) = 2(h^{1,1} - h^{2,1})$, where $h^{1,1}$ ($h^{2,1}$) is the number of (1,1) ((2,1)) forms invariant under all elements of Z. We then have,

$$\chi(H) = \frac{1}{|Z|} \sum_{g \in G} \chi(g, 1). \tag{102}$$

In all cases where a procedure for resolving the singularities is known, eq. (101–102) can be seen to give the correct answers. Later we will see such examples.

If a string theory for a manifold is known, these equations can be compared to the number of generations in this string theory, before and after a projection by some symmetry group. Consider, for example, the quintic hypersurface M_3. String propagation on this manifold is described by the theory 3^5. Concentrate on automorphism groups which are cyclic, and involve only the phases. There are three inequivalent such automorphism groups which act trivially on the $(3,0)$ form. These are generated by the elements

$$\{0,1,2,3,4\} \qquad \{0,0,2,-1,-1\} \qquad \{0,0,0,1,-1\}, \tag{103}$$

where we denoted by $\{r_n\}$ the transformation $Z_n \to e^{2\pi i r_n/5} Z_n$. The first element generates a freely acting Z_5 group. For freely acting groups, $\chi(g,h)=0$ unless $g=1$ and $h=1$. Eq. (101–102) then gives the correct answer, $\chi(H)=\chi(\tilde{H})=\chi(G)/|Z|$.

The third group is generated by the element $\{0,0,0,1,-1\}$. The submanifold left fixed by any nontrivial group element is

$$Z_1^5+Z_2^5+Z_3^5=0, \tag{104}$$

This is a Riemman surface with genus 6 and Euler number $\chi=-10$. Substituting this into eq. (101–102) we find that the Euler number of the twisted manifold is $\chi(\tilde{H})=-88$ and for the singular manifold $\chi(H)=-48$.

As explained earlier, the automorphism group of the manifold manifests itself as discrete symmetries in the 3^5 string theory. Now, the process of taking a quotient manifold has an equivalent in the string theory. Propagating a closed string on Q/Z is equivalent to projecting out all states which are not invariant under Z; this is the closed string sector. In addition, there are winding sectors which correspond to a closed string in Q/Z, which are open strings when lifted to Q, where the end points differ by a non–trivial element of Z. By implementing the formulas of [12], the exact spectrum in these sectors can be computed. The massless spectrum of a heterotic string theory on Q/Z is then found to consist of 49 generations (27 of E_6), 5 anti–generations ($\bar{27}$ of E_6), and 220 singlets of

E_6. Of these, 25 generations, and one anti–generation come from the closed string sector, and each of the four winding sectors gives 6 generations and 1 anti–generation.

We see that indeed eqs. (101–102) give the correct results. The net number of generations in the exact spectrum is -44 corresponding to Euler number -88. In the closed string sector, we find $\chi = -48$. Again, in agreement with the topological calculation.

The real importance of eqs. (101–102) is in a different situation. Namely, when we do not have a candidate manifold for some abstract string theory. The three Euler numbers appearing in eqs. (101–102) correspond, as explained above, to the net number of generations in the original theory, the closed string sector and the quotient theory. Assume that a particular theory describes string propagation on an unknown manifold. From this assumption alone, even without having an explicit manifold candidate, we can derive powerful predictions.

For simplicity, consider the case of cyclic twists by some Z_p group, where p is a prime number. The Euler number of the fixed point set $a = \chi(g, h)$ is some fixed integer for all $g, h \in Z_p$, except for $\chi = \chi(0, 0)$ which is different. Now, in the quotient theory the net number of generations is $\frac{1}{2}\chi_t$, of which $\frac{1}{2}\chi_0$ come from the closed string sector. Using eqs. (101–102) these can be expressed as

$$\chi_t = \frac{\chi + a(p^2 - 1)}{p}, \tag{105}$$

$$\chi_0 = \frac{\chi + (p - 1)a}{p}. \tag{106}$$

From eqs. (105–106) we can compute a,

$$a = \frac{p\chi_t - \chi}{p^2 - 1} = \frac{p\chi_0 - \chi}{p - 1} = \text{integer}. \tag{107}$$

Thus starting from an arbitrary $N = 2$ string theory, we can compute χ as the number of generations appearing in the original string theory, χ_t as the number of generations of the theory twisted by Z and χ_0 as the number of generations in the closed string sector. Assuming a geometrical interpretation for the theory implies that eq. (107) holds for these three numbers. Also, each of the winding sectors must contribute exactly the same number of generations. These are all highly nontrivial relations, which can easily be checked. From the viewpoint of conformal field theory, having such relations appears utterly mysterious. The only conceivable explanation for such relations is that indeed the theory is geometrical. We have checked these relations in many examples. They always work.

We conclude that there is conclusive evidence for the following [12]: All $N = 2$ string theories describe string propagation on manifolds of vanishing first Chern class.

Perhaps most importantly, the results described above show irrefutably that Calabi–Yau compactifications exist as conformal field theories and are exact solutions of the string equations of motion, despite the breakdown of conformal invariance at the four loop level of the sigma model [29], and the possibility that non–perturbative effects on the world sheet would destabilize the vacuum. The fact that we find in the spectrum all the modes corresponding to the deformations of the complex structure and the radii, shows that for any complex structure, and radii the string theories are conformally invariant, space–time supersymmetric and fully retain the topological and geometrical properties of the field theory formulation.

.8. Yukawa couplings for any scalar field theory

Consider a two dimensional $N = 2$ scalar superfield $\Phi(z, \theta^+, \theta^-, \bar\theta^+, \bar\theta^-)$ which is chiral, $D^+\Phi = \bar D^+\Phi$ where the covariant derivatives are defined by $D^\pm = \partial_{\theta^\pm} \pm i\theta^\mp\partial_z$, $\bar D^\pm = \partial_{\bar\theta^\pm} \pm i\bar\theta^\mp\partial_{\bar z}$. A typical supersymmetric lagrangian involving such a field is given by

$$\mathcal{L} = \int \mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\bar\theta^+\mathrm{d}\theta^-\Phi\Phi^* + \int d\theta^-\mathrm{d}\bar\theta^- V(\Phi) + \text{c.c.}, \tag{108}$$

The conservation of the $U(1)$ charge implies that the potential V is of the form $V = \Phi^{k+2}$ where k is some integer. More generally, we could have a number of scalar fields of various $U(1)$ charges, such that the potential V has a $U(1)$ charge 1, and is thus a quasi–homogeneous function. As a generalization of scalar field theory for $N = 0$ and $N = 1$ minimal theories [30], Kastor et al. have noticed [31] that the equation of motion of the theory eq. (108) with the superpotential $V(\Phi) = \Phi^{k+2}$ has the same form of an operator product of the k'th $N = 2$ minimal theory. Thus, it is likely that the scalar field theory has a fixed point which is described by the minimal model. The correspondence of chiral fields in the two theories is $\Phi^l = x_l$, where x_l is the l'th chiral field in the minimal theory using the notation of section (7).

If we group together $k + 2$ copies of the k'th minimal theories, the resulting potential has the form,

$$V(\Phi_i) = \sum_{i=1}^{k+2} \Phi_i^{k+2} \tag{109}$$

Intriguingly, the potential, eq. (109), has precisely the same form as the manifold which corresponds to this string theory, as we saw in section (7). More generally [32] if V is the potential of a given collection of minimal theories, $k_1 \times k_2 \times \ldots \times k_r$, $V = \sum_i \Phi_i^{k_i+2}$ can describe the Calabi–Yau manifold,

$$V = \sum_i \Phi_i^{k_i+2} = 0 \tag{110}$$

where the Φ_i are now regarded as complex variables modulo the identification $\Phi_i \equiv w^{\frac{1}{k_i+2}} \Phi_i$ and w is an arbitrary complex constant. For $r = 5$ this is a Calabi–Yau manifold. For $r < 5$ one may add trivial Φ^2 theories. (The case of $r > 5$ seem to correspond to a number of embedding spaces.) By a case by case comparison [32] it can be seen that the resulting CY manifold has the correct Euler number to give the number of generations in the string theory.

Such scalar field theory realizations are not limited to the minimal series [33], but can describe also theories of the Kazama–Suzuki type [34]. It is, in fact, not unlikely that all $N = 2$ superconformal field theories can be realized as projections of scalar field theories.

Our aim in this section is to describe how the structure constants among the chiral fields may be computed for an arbitrary scalar field theory [33]. We then use the result to compute the Yukawa couplings of the type 27^3 in the string theory compactified on any scalar field theory. The result of the calculation is then seen to agree precisely with the field theory formula for the 27^3 Yukawa couplings for the manifold $V = 0$, where V is the superpotential. This result establishes, in particular, that indeed an $N = 2$ string theory based on any scalar field theory describes string propagation on a manifold of vanishing first Chern class, as well as a quantitative identification of the superpotential and the manifold for all the complex structures.

Consider an $N = 2$ superconformal field theory. The chiral fields in the theory obey the equation $\Delta = Q/2$ where Δ is the dimension and Q is the $U(1)$ charge. All other fields in the theory obey $\Delta > Q/2$. Let C_i and C_j be two chiral fields in the theory with charges q_i and q_j. The charge of fields appearing in the operator product of the two fields C_i and C_j is $q_i + q_j$. Consequently, the dimension of such a field, Δ is greater or equal the sum of the dimensions of C_i and C_j, $\Delta \geq \Delta_1 + \Delta_2$, with equality holding only for a chiral field. Thus we have the operator product,

$$C_i(z, \bar{z}) C_j(w, \bar{w}) = f_{ij}^k C_k(w, \bar{w}) + \text{regular terms}, \tag{111}$$

where the C_k are chiral fields of charge $q_i + q_j$ and the f's are some constants. We can thus define a product structure on the set of chiral fields as the (non–singular) limit of the operator product [33],

$$C_i \times C_j = f_{ij}^k C_k. \tag{112}$$

Since this operator product is non–singular, it follows that the product so defined is associative,

$$\sum_k f_{ij}^k f_{kl}^r = \sum_k f_{ik}^r f_{jl}^k. \tag{113}$$

In addition, we can add two chiral fields (as usual) and multiply by a complex number. It follows, that the set of chiral fields forms an associative commutative algebra over the complex numbers. (An algebra is a vector space over a field which forms a ring. The ring and vector space structures are connected together by some relations. For more detail see, for example, [35].) This algebra is graded by the $U(1)$ charge; the $U(1)$ charges add when multiplying. Since the $U(1)$ charge is at most $c/3$, the algebra is finite dimensional *. The structure constants of the algebra are the operator product coefficients of the chiral fields.

The anti–chiral fields are the complex conjugate of the chiral ones, and thus form the complex conjugate algebra with the structure constants ${f_{ij}^k}^*$. Similar algebra can be obviously defined for the fields which are left–chiral and right anti–chiral, (c, a) or their complex conjugates (a, c).

Suppose that the maximal chiral field

$$C_{\max} = e^{i(\phi+\bar{\phi})\sqrt{c/3}}, \tag{114}$$

is in the spectrum of the theory. Here ϕ and $\bar{\phi}$ stand for the left and right $U(1)$ bosons. (Equivalently, from locality, all the fields of the theory obey $q_l - q_r =$ integer where q_l and q_r are the left and right charges.) We can then define a transposition operation on the chiral fields. For a given chiral field C, define the field

$$C^t = C^* e^{i(\phi+\bar{\phi})\sqrt{c/3}}. \tag{115}$$

It can easily be checked that the field C^t is chiral. Thus the transposition operation is a one–to–one map of order two, namely, $(C^t)^t = C$, on the set of chiral

$\star$ It is interesting to note that in all the known $N = 2$ conformal field theories the $U(1)$ charges are rational. This is certainly true for scalar field theories and their projections. We conjecture that it is true in general.

fields, which takes an element with the charge q to an element with the charge $c/3 - q$. (This map is not an algebra automorphism though, since it does not preserve the product structure.) This implies, in particular, that the dimension of the vector subspace of chiral fields with dimension q and $c/3 - q$ is the same.

Consider now a scalar field theory with the superpotential $V(\Phi_1, \Phi_2, \ldots, \Phi_n)$. The fields Φ_i are some chiral fields in the theory. We can form the product (in the above sense) of, say, the fields Φ_1 and Φ_2. This is a chiral field which we denote by $:\Phi_1\Phi_2:$. Continuing, in this fashion, we may form, by induction, the field $:\Phi_1^{m_1}\Phi_2^{m_2}\ldots\Phi_n^{m_n}:$ defined as the chiral field obtained by repeatedly multiplying fields. It is absolutely crucial that the algebra of chiral fields is associative and commutative. Otherwise, we would get different answers by changing the order of the multiplications. Thus we have an algebra homomorphism from the algebra of polynomials in n variables $P[x_1, x_2, \ldots, x_n]$ into the algebra of the chiral fields. The map, τ, simply takes the polynomial $x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ to the normal ordered field $:\Phi_1^{m_1}\Phi_2^{m_2}\ldots\Phi_n^{m_n}:$. This map is an algebra homomorphism (i.e., preserves the addition, product and multiplication by a complex number) as is clear from the associativity and commutativity. So far the discussion was general to any $N = 2$ theory, without assuming that it is a scalar field theory. In a scalar field theory, we expect all the chiral fields to be given in this fashion, since the fields in the theory are composite operators of the Φ's and its derivatives, but chiral fields cannot contain derivatives. Thus, in a scalar theory the map τ is onto. The kernel of map τ, $\mathrm{Ker}(\tau)$, consists of all the polynomials p for which $\tau(p) = 0$. What is $\mathrm{Ker}(\tau)$? From the equations of motion,

$$D^+\bar{D}^+\Phi_i = \frac{\partial V}{\partial \Phi_i}, \tag{116}$$

it is clear that the normal ordered fields given by the derivatives of the potential vanish. Similarly, any polynomial containing such derivatives vanishes. Thus, $\mathrm{Ker}(\tau)$ is the ideal generated by the derivatives of the potential,

$$\mathrm{Ker}(\tau) = (\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots \frac{\partial V}{\partial x_n}). \tag{117}$$

It follows (from the first isomorphism theorem) that the algebra of chiral field is isomorphic to the algebra $P[x_1, x_2, \ldots, x_n]/\mathrm{Ker}(\tau)$, the algebra of polynomials in n variables modulo the ideal generated by the derivatives of the potential. Most crucially, the structure constants among the chiral fields are given by the product of polynomials.

For example, consider the simplest potential $V(\Phi) = \Phi^{k+2}$. The chiral fields are given by Φ^l where $0 \le l \le k$. The operator product of two such fields is

$$:\!\Phi^l\!:(z):\!\Phi^m(w)\!: = :\!\Phi^{l+m}(w)\!: + \text{h.o.t} \tag{118}$$

where the coefficient in front of Φ^{l+m} is one. Using the relation to the k'th $N = 2$ minimal model, $\Phi^l = k_l x_l$ where the k_l's are some normalizations, implies

$$x_l(z) x_m(w) = C_{l,m} x_{l+m} + \ldots, \tag{119}$$

where the structure constants $C_{l,m}$ obey,

$$C_{l,m} = \frac{k_l k_m}{k_{l+m}}. \tag{120}$$

Eq. (120) is a highly non–trivial relation for the structure constants of the $N = 2$ minimal theory. The structure constants $C_{l,m}$ may be computed directly in the minimal model (e.g., ref. [36].) Indeed, they turn out to be of the form eq. (120) with the normalizations,

$$k_l = \frac{\Gamma(\frac{1}{k+2})\Gamma(1 - \frac{l+1}{k+2})}{\Gamma(1 - \frac{1}{k+2})\Gamma(\frac{l+1}{k+2})}. \tag{121}$$

There are in general many different ways to write the maximal chiral field $C_{\max}$. Since it is the unique chiral field in the theory with charge $c/3$, any polynomial with this charge has to be proportional to it. The constant of proportionality for each polynomial may be computed, up to one overall constant, using the equations $:\!p_i \frac{\partial V}{\partial \Phi_i}\!: = 0$.

From this description of the operator algebra we can compute also the transpose element, C^t, for a given chiral field C. This field may be defined as the unique chiral field which obeys the operator product $C^tC = C_{\rm max}$. Thus to compute this field with the correct normalization we simply take any maximal polynomial which contains C and divide it by the polynomial which represents C, $C^t = C_{\rm max}/C$.

For any three chiral fields C_i, C_j and C_k, consider the following structure constant,

$$C_iC_j = f_{ijk}C^t_k. \tag{122}$$

From the description of the operator algebra given above in terms of polynomials we may compute the structure constant f_{ijk}. The result is as follows. Form the product polynomial $C_iC_jC_k$. If the total $U(1)$ charge of the product is not equal to $c/3$, the structure constant f_{ijk} vanishes. If the $U(1)$ charge is equal to $c/3$ then the product polynomial is proportional to the maximal polynomial. The proportionality constant is the structure constant f_{ijk}, $C_iC_jC_k = f_{ijk}C_{\rm max}$. This proportionality constant may be computed as explained above.

Suppose that the scalar field theory under discussion has a central charge $c = 9$. We may then form a supersymmetric heterotic string theory by following the procedure described in sections (2-3). At the massless state sector of the theory we then encounter a number of space time spinor multiplets in the 27 representation of E_6 (generations), spinor multiplets in the $\bar{27}$ of E_6 (anti–generations) and spinors which are singlets of E_6. As discussed in sections (2-3), the generations correspond to fields in the internal theory with the dimension 1/2 and left and right $U(1)$ charges equal to 1. In particular, all chiral fields in the original scalar theory, with $U(1)$ charge equal to 1 would give rise to some generations. In general there might be more generations coming from other fields in the theory, namely, fields appearing as a result of the general supersymmetry projection.

Denote by ϕ and $\bar{\phi}$ the left and right moving $U(1)$ free bosons. Let C^i be any chiral field in the theory with charge one. The vertex operator for a generation which is a space–time scalar and a singlet of $SO(10)$ is

$$\Phi_0^l(z, \bar{z}) = C^l e^{-i\sqrt{3}\bar{\phi}}. \tag{123}$$

The vertex operator for a space–time fermion in the vector representation of $SO(10)$ can be written as,

$$\Psi_v^j = C^j e^{-\frac{i\sqrt{3}}{2}\phi} S_\alpha V_a, \tag{124}$$

where S_α is a space–time spinor and V_a represents the vector of $SO(10)$. In addition, in the covariant gauge, there are ghost factors.

Consider now the Yukawa couplings of three generations to one another, 27^3. For various examples these couplings were calculated [36, 37, 25]. The 27^3 couplings may be extracted from the structure constants

$$f_{ijl} = \langle \Psi_v^i \Psi_v^j \Phi_0^l \rangle, \tag{125}$$

which can be written (up to an overall constant) as

$$f_{ijl} = \langle C^i C^j C^l e^{-i\sqrt{3}(\phi+\bar{\phi})} \rangle. \tag{126}$$

The exponential in this formula is the field $C^\dagger_{\max}$. Thus f_{ijl} is identical to the structure constant appearing in the operator product,

$$C^i C^j = f_{ijl} (C^l)^t. \tag{127}$$

This is precisely the structure constant we computed earlier, eq. (122) for any scalar field theory. We conclude that the Yukawa couplings for three generations which come from the chiral fields of the scalar field theory are given as follows: simply multiply the corresponding polynomials. This is then proportional to the maximal chiral field $C_{\max}$. The proportionality constant is the Yukawa coupling, which can be computed using the equations $:p_i \frac{\partial V}{\partial \phi_i}: = 0$.

Let us consider now the field theory limit of a heterotic string propagating on the manifold M,

$$V(\Phi_i) = 0, \tag{128}$$

where Φ_i are now considered to be complex variables living in a weighted projective space defined by the quotient of C^n with the identification of points $\{\Phi_i\} = \{\alpha^{q_i}\Phi_i\}$, where q_i is the $U(1)$ charge of the field Φ_i and α is an arbitrary complex number. Assume also that the theory has the trace anomaly $c = 9$ and that there are five generating chiral fields in the potential V. If the number of generators is less then five than we add to the theory trivial Φ^2 theories, which do not change the conformal field theory, so as to make the number of generators equal to five. The manifold M is then a Calabi–Yau manifold.

The generations on the manifold correspond to the cohomology group of anti–holomorphic one forms with values in the tangent bundle, $H^1(T)$, whose elements we denote a^μ, where μ is a tangent bundle index. These one forms correspond also to the polynomial perturbations of the defining equation for the surface, eq. (128). For each polynomial that may be added to eq. (128), p, there is a corresponding one form

$$a^\mu = p\chi^\mu_{\bar\beta}\mathrm{d}x^{\bar\beta}, \tag{129}$$

where $\chi^\mu_{\bar\beta}$ is the extrinsic curvature of the manifold M. The forms so defined are closed, $\mathrm{d}a^\mu = 0$. Some of the polynomials which may be added to eq. (128) do not give a different surface, but rather can be absorbed in a redefinition of the variables ϕ_i. These polynomials are of the form $p_i\frac{\partial V}{\partial\phi_i}$ where p_i are any polynomials. These polynomials give rise to exact one forms. Thus the non–trivial perturbations of the surface M correspond precisely to the elements of the cohomology group $H^1(T)$. To summarize, the generations of the theory correspond to the elements of the cohomology $H^1(T)$ which are described as all the polynomials $p(\Phi_i)$ with $U(1)$ charge one (so they have the same weight as V)

modulo the ideal generated by the polynomials $\frac{\partial V}{\partial \Phi_i}$ ⋆ .

However, this is also precisely the space of all chiral fields in the scalar field theory with the potential V which have a charge one. To each polynomial perturbation of eq. (128) we associate the normal ordered field $:p(\Phi_i):$. As discussed earlier, in the normal ordering prescription any polynomial field containing the derivatives of the potential vanishes. Thus the chiral fields and the perturbations of eq. (128) are given by precisely the same polynomials. In particular the number of chiral fields and the number of elements of $H^1(T)$ is the same, and the generations in the string theory, based on the scalar field theory, transform under any discrete symmetry in precisely the same way as the elements of $H^1(T)$. As mentioned earlier there might be more generations in the conformal field theory which do not correspond to chiral fields in the unprojected theory. There might also be more elements of $H^1(T)$ which result from the fact that the manifold M might be singular requiring a resolution which would result in additional generations. As we saw earlier, these extra generations precisely match each other. Our general discussion, however, does not apply to these.

Consider now the Yukawa couplings of three generations. In the field theory the coupling of three generations, which are represented by the one forms a_i^μ, a_j^μ and a_l^μ is given by [39]

$$f_{ijl} = \int_M \Omega \wedge a_i^\mu \wedge a_j^\rho \wedge a_l^\nu \Omega_{\mu\rho\nu}, \tag{130}$$

where Ω is the holomorphic $(3,0)$ form. In terms of the polynomials, the Yukawa couplings are computed as follows. One multiplies the three polynomials, p_i, p_j and p_l. The product of the three polynomials is then proportional to a unique polynomial modulo the exact forms. The proportionality constant is the Yukawa coupling f_{ijl}.

⋆ Interestingly, a very similar result appears in the work of Arnold [38] as noted in [32]. Namely, the parameters of the potential which do not change the singularity type are given by a similar expression.

But this is precisely the result we found in the conformal field theory calculation of the Yukawa couplings! We conclude that the generation structure and Yukawa couplings among them are identical to the ones expected from the geometry of the manifold. This result was proved in complete generality for any scalar field theory, and applies also to non–rational theories. We are bound to conclude that the string theory based on any scalar field theory with an arbitrary potential V corresponds to string propagation on the manifold $V(\Phi_i)=0$.

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